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PREFACE TO THE FOURTH EDITION

THE reader's attention is called to several changes in the arrangement of the material and to some additions to the subject matter in this fourth edition of *Applied Mechanics*. On account of the fact that coplanar force systems are usually simpler of solution than noncoplanar systems, they are all given first in their order of simplicity.

In order to give better continuity, nearly all the kinematic or pure-motion discussion has been placed together in one chapter. For the same reason, it was considered advisable to place all the material on the subject of work and energy in one chapter, except as the principles of work and energy are used in the later parts of the book.

In conformity with present practice in structural work, loads on structures are given in kips instead of in pounds as in previous editions.

Some new material has been added in the chapter on moment of inertia of mass. The moment of inertia of thin plates has been given, including the subject of principal axes and the ellipse of inertia.

The data of all problems have been made new, and in many places new problems drawn from the expanding field of engineering have been added. As in previous editions, answers to all problems have been given.

In conclusion, the author wishes to thank all his friends and especially his colleagues at Purdue for their valuable suggestions in regard to subject matter and methods of treatment and arrangement. In particular, he wishes to thank Mr. P. H. Schneck, one of his former students, who is now with the Curtiss-Wright Corporation, for data on weights, dimensions, material, and moments of inertia of airplane propellers and rotating assemblies.

A. P. POORMAN.

PURDUE UNIVERSITY,
March, 1940.

PREFACE TO THE FIRST EDIT...

THIS text-book on Applied Mechanics is inten... undergraduate courses in Mechanics in engineerin... knowledge of the principles of General Physic... d is assumed. The work in its present form gre... attempt to develop the basic principles of the s... which the average student could easily follow and ... illustrations as would show clearly the application o... ciples to the solution of engineering problems.

Two features may be pointed out in which a departure f... the usual procedure has been made which it is hoped will be advantageous to the student. One of these is the extended use whic... has been made of the graphic method of solution. It has been th... author's experience that the graphic method is valuable not o... on account of the ease and rapidity with which it may be app... to the solution of certain classes of problems, but also on ac... of the aid it gives in understanding the algebraic method ... principles underlying the two methods are developed coör... in order to show their relation. The graphic meth... used wherever its application tends to promote clearness.

The other special feature is the large numbe... strative examples which have been solved in detail to sh... relation between the principle which has been developed a... the problems to which it applies.

More problems are included than can usually be assigned if the book is to be completed in one semester. Those included in the articles should always be solved; the general problems at the end of each chapter may be used as the instructor prefers. The answers ... are given, since it has been found that the ... rk at a problem with more interest if he ... check his result. Those instructors

who prefer no answers to be given may make suitable changes in the data of the problems.

In conclusion the author wishes to thank his colleague, Professor Richard G. Dukes, for his careful reading of the manuscript and for his helpful suggestions in regard to form and content.

A. P. POORMAN.

PURDUE UNIVERSITY,
March, 1917.

CONTENTS

PART II. DYNAMICS

PAGE

CHAPTER XI. KINEMATICS OF A PARTICLE

CHAPTER XII. KINETICS OF RIGID BODIES IN RECTILINEAR TRANSLATION

CHAPTER XIII. MOMENT OF INERTIA OF MASSES

CHAPTER XIV. ROTATION OF RIGID BODIES

APPLIED MECHANICS

PART I. STATICS

CHAPTER I

DEFINITIONS AND GENERAL PRINCIPLES

1. Definitions.—*Mechanics*, in general, is that branch of physical science which treats of the effects of forces upon the motion or upon the condition of material bodies. It includes physics, celestial mechanics, fluid mechanics, applied mechanics, and strength of materials.

Applied mechanics, as treated in this book, des the laws of mechanics which are applicable to the study of the motions of particles and of rigid bodies as used in problems in engineering. The condition of rest is considered to be the limiting condition of motion.

A particle is a body or a part of a body the dimensions of which are so small as to be negligible when compared with its surroundings or with its range of motion, so that the forces acting upon it may be considered to be localized at a point. A rigid body is a collection of material particles the distances between which do not change. Physical bodies are not absolutely rigid; but when an analysis is made of their behavior in certain ses, such as equilibrium, their changes in shape and size may be neglected without appreciable error.

The subject of applied mechanics may be divided into two parts, statics and dynamics, and dynamics may be further divided into kinematics and kinetics. Statics is that part of the subject which treats of bodies in equilibrium. Such bodies are usually at rest with respect to some base of reference but may have a constant velocity with respect to this base. Dynamics is that part of the subject which treats of particles and bodies that are in motion with respect to some base of reference. *Kine-*

matics is that part of dynamics which treats of the motion of particles and rigid bodies without reference to the forces that produce or change the motions. *Kinetics* is that part of dynamics which treats of the motions of material bodies as caused or changed by the application of forces. The limiting case of bodies with constant velocity and therefore in equilibrium is sometimes treated in kinetics as well as in statics.

2. Fundamental Quantities and Their Units.—Time, space, force, and mass are the fundamental quantities used in mechanics. Since time, space, force, and mass are elemental concepts, they are incapable of definition in terms of each other or of anything simpler. It is assumed that they are apprehended by the student from his previous experience.

Each of these fundamental quantities is capable of measurement by the simple process of comparing it with a standard quantity of the same kind which has been selected arbitrarily as the unit.

There are two systems of units in common use, the centimeter-gram-second (c.g.s.) system and the foot-pound-second (f.p.s.) system. The engineers of English-speaking countries use the foot-pound-second system almost exclusively, so it will be the only one used in this book.

Time is recognized by the succession of events. The units of time commonly used are the second and its multiples, the minute and hour.

Space is recognized as extending in every direction. The units of space commonly used are the foot, the inch, and the mile.

Force is recognized by its tendency to change the motion, size, or shape of bodies of matter. A *force* may be defined as an action of one body upon another body which changes or tends to change the state of motion of the body acted upon. It is seen from this that forces always occur in pairs, but usually only one force of the pair is considered. In our experience the most common force is the earth pull—the attraction of the earth for all bodies upon it. This pull of the earth upon bodies is called their *weight* and is measured by means of the spring balance. Weight varies slightly with the latitude and with the height above sea level, but this variation is so small that it may be neglected in practically all problems in engineering. The unit of force commonly used in engineering work is the pound,

the amount of the earth pull upon the "standard pound" kept at the Bureau of Standards, Washington, D. C.

In some problems in which large forces are considered, the larger units of the *kip* (1000 pounds) or the *ton* (2000 pounds) are used. Loads on structures are usually given in kips. Bearing pressures of soils under foundations are usually given in tons per square foot.

Mass, or *matter*, is recognized as something occupying space. The mass of a body is the quantity of matter in it and is constant regardless of position. Mass is usually determined by means of the lever-arm balance. Since g, the acceleration of a freely falling body, varies with the latitude and with the height above sea level the same as the weight W varies, the mass of a body may also be determined from the ratio of simultaneous values of W and g, or

$$M = \frac{W}{g}$$

The unit of mass used in engineering is a derived unit and is the amount of matter that will be given unit acceleration by unit force. The discussion of the unit of mass will be taken up more fully in Chap. XII.

3. Classification of Forces.—Forces may be classified as concentrated forces and distributed forces. A *distributed* force is one whose place of application is an area. A *concentrated* force is one whose place of application is so small that it may be considered to be a point. In many cases a distributed force may be considered as though it were a concentrated force acting at the center of the area of application or at the center of the force system.

Forces are sometimes classified as forces at a distance and forces by contact. Magnetic, electrical, and gravitational forces are examples of forces at a distance. Gravitational force, or the weight of bodies, is the chief one considered in applied mechanics. The pressure of steam in a cylinder and that of the wheels of a locomotive on the supporting rails are examples of forces by contact.

4. Scalar Quantities and Vector Quantities.—A *scalar* quantity is a quantity that has magnitude only. Volumes and masses are examples of scalar quantities.

A *vector* quantity is a quantity that has direction as well as magnitude. Forces, velocities, accelerations, and momenta are examples of vector quantities. Vector quantities are represented by lines called *vectors*, which have definite length and direction. For any given vector quantity, the length of the vector represents to some scale the magnitude of the quantity, and an arrowhead shows the direction.

In Fig. 1, vector a represents a velocity downward of 16 feet per second; vector b represents a velocity upward of 12 feet per second; and vector c represents a velocity to the left of 6 feet per second. In Fig. 2, vector a_n represents an acceleration toward the center O of 80 feet per second per second, and a_t represents an acceleration tangent to the circle of 20 feet per second per second.

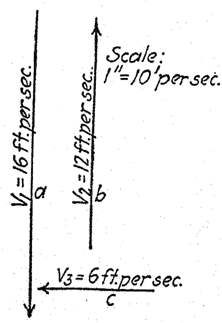


FIG. 1.

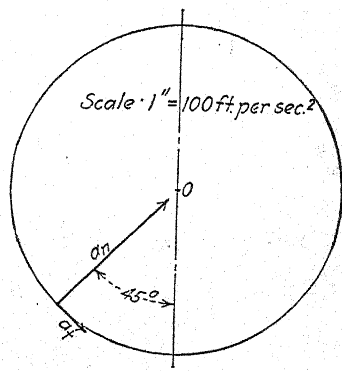


FIG. 2.

Forces are vector quantities that have position or line of action in addition to magnitude and direction. In any space diagram, the force vectors must be shown in their proper position, as in Fig. 3(*b*).

5. Three Methods of Analysis of Problems.—A *problem* in mechanics consists of a statement of certain known quantities and relations from which certain other unknown quantities or relations are to be determined.

There are three common methods of analysis of problems, the graphic method, the trigonometric method, and the algebraic method. In the graphic method, the quantities are represented by corresponding lines or areas; the relations between them are represented by the relations of the parts of the diagram; solution

is made by completing the diagram according to the known relations; and the amounts of the unknown quantities are determined by the measurement of resulting lines, areas, or angles.

In the trigonometric method, the quantities are represented by lines or areas, as in the graphic method, except that they are not necessarily drawn to scale. Solution is made from the known trigonometric or geometric relations between the parts of the diagram.

In the algebraic method, quantities are represented by symbols; the relations between them are shown by signs indicating operations; and the solution of the resulting equations is made by algebra, arithmetic, and the calculus.

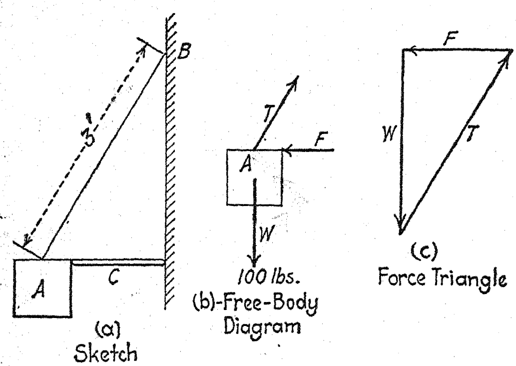


FIG. 3.

All three of these methods will be used in the solution of problems in this book. The student should develop proficiency in the three methods equally and likewise the ability to select the method best suited to the solution of any given problem.

6. Free-body Diagram.—The *free-body method* of analyzing problems will be used chiefly in this book. In applying this method, the whole body or some part of it is considered as separated from the surrounding parts. This "free body" is represented diagrammatically, with the actions upon it of the parts removed indicated by vectors, known or unknown. From the conditions and forces that are known, the unknown relations and forces are determined.

Figure 3(a) is a sketch drawn to scale, representing a cubical block A, 8 inches square, weighing 100 pounds, suspended from point B on a vertical wall by a cord 3 feet long and held away from

the wall by a rod C 14 inches long. Figure 3(b) is the free-body diagram, in which the block A is shown "free" from its surroundings. The force of its weight is shown as vector W, the earth pull of 100 pounds. The pull in the cord, upward and to the right, is shown as vector T; and the horizontal thrust of the short rod C is shown as vector F. In this diagram, vectors T and F are unknown in amount and cannot be drawn to scale; therefore it is not necessary to draw the known vector W to scale. Figure 3(c) shows the force triangle in which solution has been made for the unknown forces T and F according to the principles of Art. 19. In this diagram, the forces are all drawn to scale.

The student is advised that it is important to develop early the ability to draw the sketch and the free-body diagram for any given problem, because the free-body diagram correctly drawn constitutes one of the chief steps in the solution.

It should be noted that the equal and opposite forces that the free body exerts upon the earth, the cord, and the short rod are not considered. The forces considered are those which other bodies exert upon the free body.

7. Transmissibility of Forces.—It has been found from experience that the external effect of a force upon a rigid body is the same for all possible points of application along the line of action of the force. It follows, then, that in the solution of problems a force may be considered to be transferred either backward or forward along its line of action.

The internal effects of a force—the stress and deformation caused by it—will of course differ for different points of application of the force along its line of action.

8. Homogeneous Equations.—An algebraic equation stating the relation between physical quantities must be homogeneous; that is, it must be expressed in corresponding units. This is necessary in order that when the units are supplied, the operations of addition and subtraction may be performed upon quantities of the same kind, and so that the quantities on the two sides of an equation may have the same dimension.

As an example, consider the equation derived in physics for the distance traveled by a falling body projected downward with an initial velocity v_o. The equation is $s = v_o t + \frac{1}{2}gt^2$, in which s is the distance traveled, g is the acceleration of gravity, and t is the time. These quantities must be given in corresponding

units. If the quantity g is used in feet per second per second, v_o must be in feet per second, t must be in seconds, and s is then given in feet.. The dimensional equation in which each quantity is replaced by its unit becomes

$$\text{Feet} = \frac{\text{feet}}{\text{seconds}} \times \text{seconds} + \frac{\text{feet}}{\text{seconds}^2} \times \text{seconds}^2$$

The "seconds" in the two terms on the right-hand side of the equation cancel each other, and the resulting expression is

$$\text{Feet} = \text{feet} + \text{feet}$$

If it were desirable to use the velocity v_o in units of miles per hour, it would be necessary to use the acceleration g in miles per hour per hour, and t in hours. The quantity s would then be given in miles.

9. Solution of Problems.—In order to understand thoroughly such a subject as applied mechanics, it is necessary for the student to solve a generous number of problems. Whenever the application of the theory to the solution of a problem demands it, examples are solved for the purpose of illustration. These should always be studied thoroughly by the student, and all new points should be noted carefully. It will aid in the mastery of the principle if the book is then laid aside and the example solved independently before beginning the solution of the regular problems.

The problems at the end of each article are, in general, direct applications of the principle or principles developed in that article, and the solution should always be made accordingly. The general problems at the end of each chapter often involve the principles developed in several different articles. In making solution, the student often has a choice of two or more methods, and in such cases the method best adapted to the problem should be used. If time permits, a check solution by another method will give confidence in the results obtained.

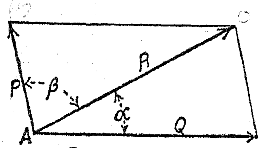
In case there is no figure in the text to illustrate a problem, a sketch properly drawn is a great help in the solution. In simple problems, this sketch can be made into the free-body diagram; in more complex problems, separate diagrams should be used.

The free-body diagram should always be drawn.

CHAPTER II

COPLANAR, CONCURRENT FORCES

10. Definition.—A system of forces is composed of two or more forces, all of which are acting upon the same free body. A *coplanar* system of forces is one in which all the forces of the system are in the same plane. A *concurrent* system of forces is one in which the lines of action of all the forces pass through a common point.

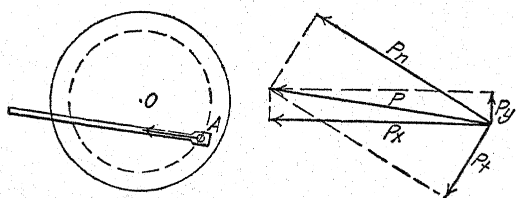


Space and Force Diagrams Combined

FIG. 4.

In this chapter, the term "force system" will include only coplanar, concurrent force systems.

11. Graphical Representation of Forces.—As stated in Art. 4, forces can be represented graphically by vectors. For comparatively simple problems, only one diagram is used, in which



(a) Space Diagram

(b) Force Diagram

FIG. 5.

case each vector shows the line of action, direction, and magnitude of the force represented, as in Fig. 4. For more complicated problems, two diagrams are used, the space diagram showing the lines of action of the forces, and the force diagram showing the magnitude of the forces, as in Fig. 5. The direction of the forces may be shown in either diagram but is usually shown in both.

Problems

1. In Fig. 4, determine the values of forces P, Q, and R if 1 in. = 100 lb. Scale the angles α and β. *Ans.* 60 lb.; 116 lb.; 118 lb.

2. If in Fig. 5 the value of force P is 3000 lb., what are the scaled values of P_x, P_y, P_n, and P_t? *Ans.* 2950 lb.; 435 lb.; 2760 lb.; 1200 lb.

12. Resultant of Two Forces, Graphically. *The Parallelogram Law.*—If two concurrent forces are represented by their vectors, both of which are directed either toward or away from their point of intersection, the diagonal of the completed parallelogram drawn through their point of intersection represents their resultant. In Fig. 6, let vectors MN and KL represent two forces whose lines of action intersect at O. By the principle of Art. 7, the forces may be transmitted along their lines of action until they are in the positions OA and OB. Line AC is drawn parallel to OB, and line BC parallel to OA, to complete the parallelogram $OACB$. The diagonal OC is the vector sum of the two vectors OA and OB and represents the resultant of the two forces.

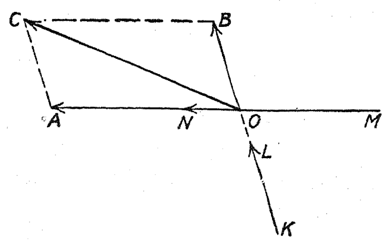


FIG. 6.

If the two vectors had been placed so that they were both directed toward their point of intersection, their resultant vector would have been the same, in amount, direction, and line of action.

The Triangle Law.—If two concurrent forces are represented by their vectors laid down in order as the two sides of a triangle, the third side of the triangle drawn from the initial point to the final point represents their resultant. Figure 7 shows the two possible solutions for obtaining the resultant of the forces KL and MN of Fig. 6. In the lower triangle, vectors OA and AC represent the two forces, and vector OC represents their resultant. In the upper triangle, vectors OB and BC represent the two forces, and vector OC represents their resultant as before.

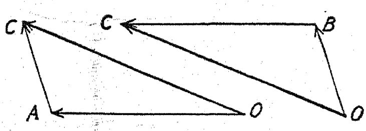


FIG. 7.

These diagrams are *force diagrams only.* The line of action of the resultant must pass through the point of intersection of the two lines of action in the space diagram.

If two forces have the same line of action and the same direction, the force triangle becomes a continuous straight line, and

the vector sum of the two components is the same numerically as their arithmetic sum. If they have the same line of action and are opposite in direction, the force triangle is again a straight line, and the vector sum of the two components is the same numerically as their arithmetic difference.

In case the two forces to be combined are so nearly parallel that their lines of action do not meet on the diagram, as forces P and Q (Fig. 8), the amount and direction of their resultant R could be obtained by the triangle law, but not its position. To locate

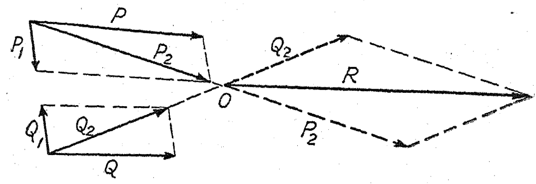


FIG. 8.

the position of the resultant, a modification of the parallelogram law is very useful. Any two equal, opposite collinear forces P_1 and Q_1 may be added to the system without change of effect. At their point of concurrence, forces P and P_1 are combined into their resultant P_2, and likewise forces Q and Q_1 into their resultant Q_2. These two resultant forces of the system are now concurrent at O and are combined by the parallelogram law into their resultant R, which now is in its correct position in space.

If forces P and Q were extended beyond the limits of the diagram to their point of concurrence, this point would be on the line of action of the resultant R.

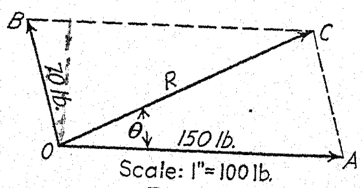


FIG. 9.

EXAMPLE

A force of 150 lb. acting horizontally to the right is to be combined with one of 70 lb. acting upward and to the left at an angle of 15° with the vertical. Solve for resultant R and angle θ between R and the 150-lb. force.

Solution.—From point O, Fig. 9, vector OA is laid off to scale, the scale chosen being 1 in. = 100 lb. From the same point O, vector OB is laid off to scale. The parallelogram is completed by drawing AC parallel to OB and BC parallel to OA. The diagonal OC represents the resultant, and scales 148 lb. The angle θ scales 27°.

It will be seen that either triangle OAC or triangle OBC will give the complete solution for the amount and direction of the resultant R.

Problems

1. Figure 10 represents a plate 6 ft. wide and 4 ft. high, in the plane of which forces are applied as indicated. Combine the 40- and the 60-lb. forces by means of the parallelogram law. Scale the angle that the resultant makes with the X axis. Suggested scale: 1 in. = 20 lb.

Ans. 87.6 lb.; 22°.

2. Combine the 40- and 20-lb. forces shown in Fig. 10 by means of the triangle law. Scale the angle that the resultant makes with the X axis. Suggested scale: 1 in. = 10 lb.

Ans. 28.6 lb.; 14°05′.

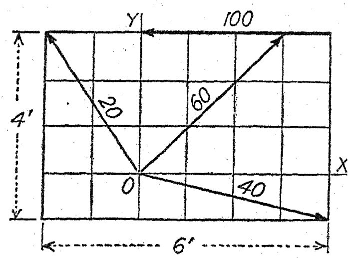


FIG. 10.

3. Combine the 40- and the 100-lb. forces shown in Fig. 10. Locate the point where the resultant intersects the Y axis. *Ans.* 62 lb.; 4.9 ft.

13. Resultant of Two Forces, Trigonometrically.—In Fig. 11, P and Q are two forces, concurrent at O, at an angle α with each other. By the cosine law from trigonometry,

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha}$$

Also,

$$\tan \theta_P = \frac{S}{P + T} = \frac{Q \sin \alpha}{P + Q \cos \alpha}; \tan \theta_Q = \frac{P \sin \alpha}{Q + P \cos \alpha}$$

In the special cases in which $\alpha = 0°$, 90°, or 180°, these expressions are very much simplified.

For $\alpha = 0°$,

$$R = P + Q, \text{ and } \theta = 0°$$

For $\alpha = 90°$,

$$R = \sqrt{P^2 + Q^2}, \text{ and } \theta = \tan^{-1}\frac{Q}{P}$$

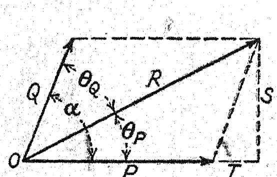


FIG. 11.

For $\alpha = 180°$,

$$R = P - Q, \text{ and } \theta = 0° \text{ or } 180°$$

EXAMPLE

In Fig. 11, let force P = 100 lb., Q = 80 lb., and angle α = 150°. Compute R and θ_P.

Solution.

$$R^2 = 10{,}000 + 6400 - 2 \times 100 \times 80 \times 0.866$$
$$R = 50.4 \text{ lb.}$$
$$\tan \theta_P = \frac{40}{100 - 69.3} = 1.303$$
$$\theta_P = 52°30'$$

Problems

1. In Fig. 11, let force $P = 520$ lb., $Q = 120$ lb., and $\alpha = 75°$. Compute R and θ_P.

Ans. 563 lb.; 11°53′.

2. Combine the 20- and the 60-lb. forces shown in Fig. 10 by the method of this article. Compute the angle that the resultant makes with the X axis.

Ans. 66.8 lb.; 62°05′.

3. Combine the forces shown in Fig. 12 into their resultant. Determine the distance from A at which the resultant intersects the base line AB.

Ans. 8.57 kips; 46.8 ft.

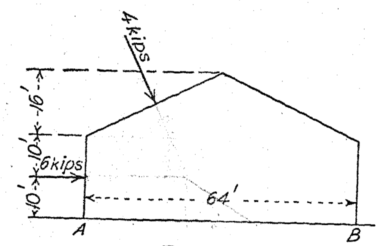


FIG. 12.

14. Resultant of Three or More Forces, Graphically.—By an extension of the triangle law of Art. 12, the resultant of any number of concurrent forces may be found. In Fig. 13, AB and BC are combined into their resultant AC; AC and CD are combined into their resultant AD; and finally AD and DE are combined into their resultant AE, which is therefore the resultant of the entire system. Its line of action passes through O, the common point of the system.

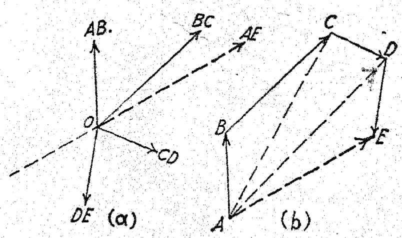


FIG. 13.

It will be noticed that in making the solution, vectors AB, BC, CD, and DE may be laid down in order; then the final resultant AE may be drawn without using the intermediate resultants AC and AD. Figure 13(*b*) is called the *force polygon.* If the forces are taken in any other order, a different force polygon will be obtained, but the same resultant.

Problems

1. In Fig. 10, combine into their resultant the three forces that are concurrent at O.

Ans. 85.8 lb.; 35°05′.

2. Combine the following forces into their resultant. The angle α is the angle between the force and the positive end of the X axis. $F_1 = 600$ lb., $\alpha_1 = 30°$; $F_2 = 400$ lb., $\alpha_2 = 225°$; $F_3 = 300$ lb., $\alpha_3 = 285°$.

Ans. 416 lb., 319°.

15. Resolution of a Force into Components.—By reversing the parallelogram law or the triangle law, any force may be resolved into two components. There are four cases, as follows:

1. Directions of both components known, to find their amounts.
2. Amount and direction of one component known, to find amount and direction of other component.
3. Amounts of both components known, to find their directions.
4. Amount of one component and direction of other known, to find direction of first component and amount of second component.

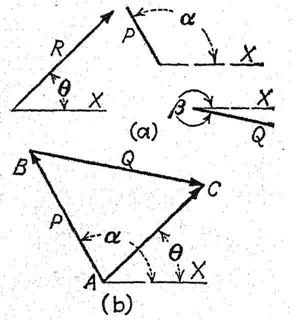


FIG. 14.

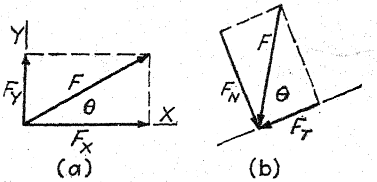


FIG. 15.

To illustrate Case 1, let R, Fig. 14, be the given force, at an angle θ with the X axis. Let it be required to resolve force R into two components, one parallel to P at an angle α with the X axis, and the other parallel to Q at an angle β with the X axis. The construction is shown in Fig. 14(*b*). Vector $AC = R$ is drawn at the angle θ with the X axis. Through point A, vector AB is drawn parallel to P. Through point C, vector BC is drawn parallel to Q. Their point of intersection B determines their amounts. If vector Q had been drawn through point A, and vector P through point C, the other triangle of the parallelogram would have been formed.

The components usually desired are those parallel to given rectangular axes, such as F_x and F_y, Fig. 15(*a*), or F_N and F_T, Fig. 15(*b*). In Fig. 15(*a*), $F_x = F \cos \theta$, and $F_y = F \sin \theta$. The components F_x and F_y are called the *projections* of the force F

upon the X and Y axes. In Fig. 15(b), $F_T = F \cos \theta$, and $F_N = F \sin \theta$.

To illustrate Case 2, let R, Fig. 16(a), be the given force, at an angle θ with the X axis, and P the amount of one of the components, at an angle α with the X axis. The solution is shown in Fig. 16(b). Vector $AC = R$ is drawn at angle θ with the X axis. Vector $AB = P$ is drawn at angle α with the X axis. Vector BC, at angle β with the X axis, must be the other component Q.

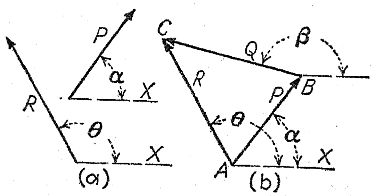


FIG. 16.

To illustrate Case 3, let R, Fig. 17(a), be the given force, and let lines P and Q represent the amounts of the required components. To make the solution, vector $AB = R$, Fig. 17(b), is drawn. With point A as a center and a radius equal to P, an arc is drawn at C. With point B as a center and a radius equal

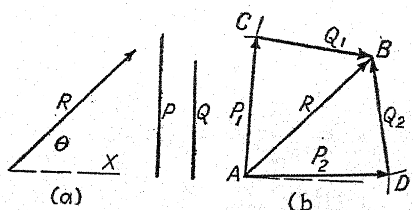


FIG. 17.

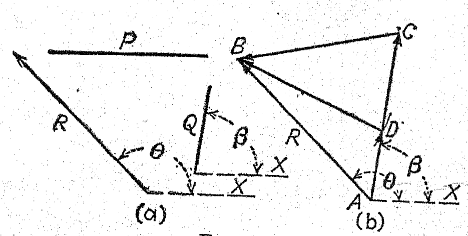


FIG. 18.

to Q, an intersecting arc is drawn, thus determining point C. Vectors P_1 and Q_1 then give one possible solution. In a similar way, vectors P_2 and Q_2, intersecting at D, give another possible solution. If the numerical sum or difference of P and Q exactly equals the amount of R, there is only one possible solution, and components P and Q act along R. If the numerical sum is less

than the amount of R, or the numerical difference is greater than the amount of R, no solution is possible.

To illustrate Case 4, let R, Fig. 18(a), be the given force. Let line P represent the amount of one component, and line Q show the direction of the other. To make the solution, vector $AB = R$, Fig. 18(b), is drawn. Through point A, line AC is drawn parallel to line Q. With point B as a center and with a radius equal to line P, arcs are drawn intersecting line AC at C and D. There are two possible solutions. Either pair of vectors, AC and CB, or AD and DB, satisfy the conditions. If the length of line P is the same as the normal distance from point B to line AC, there is only one possible solution. If the length of line P is less than the normal distance from point B to line AC, no solution is possible.

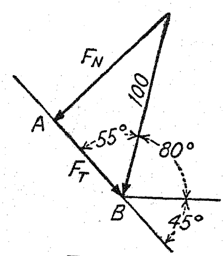


FIG. 19.

EXAMPLE

In Fig. 19, resolve the 100-lb. force into components parallel to the line AB and normal to the line AB.

Solution.—The angle between the force and line AB is 55°. The component parallel to line AB is

$$F_T = 100 \cos 55° = 57.36 \text{ lb.}$$

The component normal to line AB is

$$F_N = 100 \sin 55° = 81.92 \text{ lb.}$$

Problems

1. In Fig. 19, resolve the 100-lb. force into its horizontal and vertical components. *Ans.* 17.37 lb.; 98.48 lb.

2. In Fig. 19, resolve the 100-lb. force into a horizontal component of 60 lb. acting to the left, and another component. *Ans.* 107.4 lb., 23°30′ with vertical.

3. A vertical force of 10 lb. is to be resolved into two components, one of 20 lb., the other of 25 lb. Determine the angle of each with the horizontal. *Ans.* 18°10′; 40°30′.

4. Resolve a vertical force of 10 lb. into two components, one of which is 25 lb. in amount, the other of which is horizontal. Determine the direction of the 25-lb. component and the amount of the horizontal component. *Ans.* 23°35′; 22.9 lb.

16. Resultant of Three or More Forces, Algebraically.—By the principle of Art. 15, each force of a coplanar, concurrent system

may be resolved into its X and Y components at the point of concurrence. All the X components may be added algebraically to obtain a single X force ΣF_x, and all the Y components may be added algebraically to obtain a single Y force ΣF_y. These two components may then be combined into the final resultant R of the system.

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

EXAMPLE

Determine the amount and direction of the resultant of the four forces represented in Fig. 20.

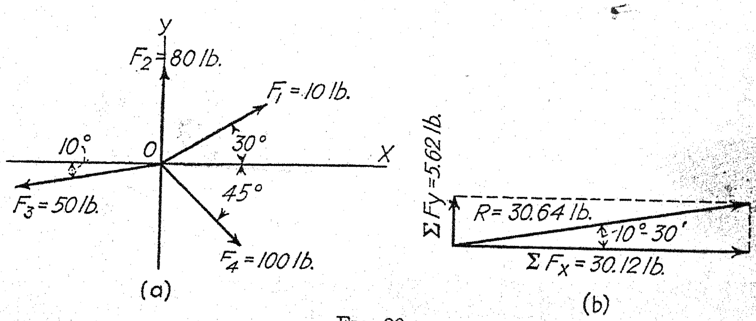


FIG. 20.

Solution.—Each force in order is replaced by its X and Y components, $F_x = F \cos \alpha$ and $F_y = F \sin \alpha$, as tabulated below.

F	F_x	F_y
10	8.66	5.
80	0.	80.
50	−49.24	− 8.68
100	70.7	−70.7
	$\Sigma F_x = +30.12$	$\Sigma F_y = +5.62$

The X components are added algebraically and give +30.12 lb. for the resultant force along the X axis, ΣF_x. In the same way, the Y components are added algebraically and give +5.62 lb. for the resultant force along the Y axis, ΣF_y. The final resultant is obtained by taking the square root of the sum of the squares of the two rectangular components.

$R = \sqrt{30.12^2 + 5.62^2} = \sqrt{938.8} = 30.64$ lb., as shown in Fig. 20(*b*). The angle θ with the X axis is given by the expression

$$\theta = \tan^{-1} \frac{\Sigma F_y}{\Sigma F_x} = \tan^{-1} \frac{5.62}{30.12} = 10°30'$$

Problems

1. Use the line of action of the 100-lb. force as the X axis, and check the result of the foregoing example.

2. Compute the resultant of the three forces shown in Fig. 21.

Ans. 801 lb., 69°05′.

17. Moment of a Force with Respect to a Point.—The moment of a force with respect to a point is the product of the force and the perpendicular distance from its line of action to the point. This perpendicular distance is called the *arm* of the force; the point is called the *center of moments*.

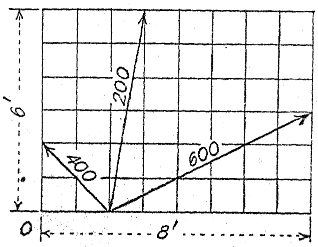


FIG. 21.

Let F be the force, and d the perpendicular distance from any point O to the line of action of the force. Then the moment of the force F with respect to point O is

$$M_o = Fd.$$

It will be seen that the moment of a force with respect to a point is the same as its moment with respect to an axis passing through the point normal to the plane through the force and the point.

Moment is measured in terms of the units of force and length used; for example, pound-foot (lb.-ft.), pound-inch (lb.-in.), kip-foot (kip-ft.).

A moment tending to produce counterclockwise rotation when viewed from the positive end of any coordinate axis is commonly called a *positive* moment, and one tending to produce clockwise rotation is called *negative*. The opposite notation may be used if kept consistently throughout the problem.

Problems

1. In Fig. 21, compute the moment of each of the three forces with respect to point O.

Ans. M_o of 400 lb. = +565.6 lb.-ft.; M_o of 200 lb. = +394.6 lb.-ft.; M_o of 600 lb. = +536 lb.-ft.

2. In Fig. 20, compute the moment of the four forces with respect to a point on the X axis 6 in. to the right of point O. *Ans.* −33.7 lb.-in.

18. Principle of Moments: Varignon's Theorem.—*To prove that the algebraic sum of the moments of two concurrent forces with*

respect to a point in their plane is equal to the moment of their resultant with respect to the same point.

Let P and Q, Fig. 22, be the forces concurrent at A, R their resultant, and O any point in their plane. Draw AO, and produce it to F. From the ends of P and R, drop perpendiculars CE, DF, and CG. Also drop perpendiculars p, q, and r from O to the forces P, Q, and R, respectively. Let α, β, and θ be the angles between the line AO and the forces P, Q, and R, respectively. Then

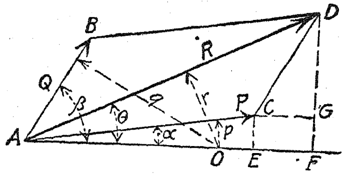


FIG. 22.

$$\overline{FD} = \overline{FG} + \overline{GD}$$

so

$$R \sin \theta = P \sin \alpha + Q \sin \beta$$

This equation multiplied by $\overline{OA}$ becomes

$$R \cdot \overline{OA} \sin \theta = P \cdot \overline{OA} \sin \alpha + Q \cdot \overline{OA} \sin \beta$$

so

$$Rr = Pp + Qq$$

Since Rr is the moment of the resultant R with respect to point O, and Pp and Qq are the moments of the forces P and Q, respectively, about the same point, the principle stated above is proved.

In computing the moment of a given force with respect to a point, it is often simpler to resolve the force into components and compute the moment of these components rather than to compute the lever arm of the force.

The proof given above may be extended to the case of three or more concurrent forces. The statement of the general case, then, is as follows: The moment of the resultant of any number of concurrent forces in a plane with respect to any point in that plane is equal to the algebraic sum of the moments of the forces with respect to the same point.

Problems

1. Assuming that the resultant of the three forces shown in Fig. 21 is 801 lb. at an angle of 69°05′ with the X axis, as given in Prob. 2, Art. 16, compute the moment of this resultant with respect to point O. Check with

the algebraic sum of the moments of the three components as given in Prob. 1, Art. 17. *Ans.* 1497.2 lb.-ft.

2. Outline the method of proof for Varignon's theorem if point O is between force P and the resultant R.

19. Equilibrium of Three or More Forces: Graphic Solution.—If in a system of forces such as that represented in Fig. 13 the vector of the last force closes at the starting point, the resultant $R = 0$, and the system is in equilibrium. In any case, another force equal and opposite to R through the common point of the system will hold it in equilibrium.

Conversely: If the force system is in equilibrium, the force polygon must close.

The most common case is that in which two of the forces are unknown in amount but known in direction. In the solution, vectors to represent the known forces are laid down in any order "head to tail." Then a pair of vectors drawn parallel to the two unknown forces, one through the final point of the last known vector and the other through the initial point of the first vector, will complete the force polygon. The scaled values of these two vectors give the two unknown quantities.

If one of the forces is unknown both in amount and direction, a vector from the final point of the last known vector to the initial point of the first vector will determine both the unknown elements.

If two of the forces are unknown in direction but known in amount, the solution is made by using the known vectors as radii, one with its center at the initial point, the other with its center at the final point, and drawing intersecting arcs. The directions of the two closing forces are determined by this point of intersection. In general, two solutions are possible.

Since for the closing of the force polygon there are only two conditions, it is evident that no more than two unknown elements can be determined. If for any given coplanar, concurrent system there are three unknown elements, such as the amounts of three forces or the amount of one force and the amount and direction of another, the problem cannot be solved.

In the special case of three forces in equilibrium, the following important principle also applies:

If a force system of three nonparallel forces is in equilibrium, they must meet in a common point.

For, if any two of the forces are combined into their resultant, this resultant acts through their point of intersection. Then in order for the third force to balance this resultant and hence the other two forces, it must also pass through their point of intersection and be equal and opposite to their resultant.

EXAMPLE

A block weighing 50 lb. is supported on a 30° plane as shown in Fig. 23. If the friction of motion is 15 lb., what force P parallel to the plane will be required to move the block at a uniform speed up the plane? What is the normal pressure of the plane on the block?

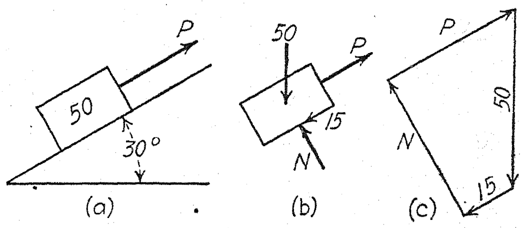


FIG. 23.

Solution.—The sketch, Fig. 23(a), shows the block resting on the plane with the force P acting upon it. In the free-body diagram, Fig. 23(b), the actions of all the surrounding parts upon the block are shown as vectors. The weight is a force of 50 lb. vertically downward; the friction is a force of 15 lb. acting downward parallel to the plane; the reaction of the plane normal to the surface of contact is force N, acting upward; and force P acting upward parallel to the plane is the force required. Forces N and P are known in direction but unknown in amount.

The block is to be moving at a constant speed, so the forces are in equilibrium, and the force polygon must close. In Fig. 23(c), the 50-lb. vector representing the weight is drawn to scale; then, from the head of it, the 15-lb. vector representing the frictional force is drawn. From the head of this vector, the vectors N and P in their known directions must close at the initial point. Vector N scales 43.3 lb., and vector P scales 40 lb.

Problems

1. If, in the example above, the friction of motion is 30 lb., solve for force P parallel to the plane to move the block at a constant speed down the plane. *Ans.* 5 lb.

2. A weight of 600 lb. is suspended by a cable 80 ft. long. A diagonal pull at an angle of 45° with the horizontal holds the weight 20 ft. from the vertical line through the support. Solve for this diagonal pull and the tension in the cable. *Ans.* 295 lb.; 835 lb.

3. Figure 24 represents a weight of 800 lb. supported by two cords, one 12 ft. long, the other 20 ft. long, with points of support 30 ft. apart. Solve for the tensions T_1 and T_2 in the cords. *Ans.* 1150 lb.; 1065 lb.

4. Figure 25 represents the left pedestal of the truss of a railroad bridge. The end reaction is 206 kips. Solve for the compression in the end chord AB and the tension in the lower chord BC. *Ans.* 291.3 kips; 206 kips.

5. Solve for the tension in each cord and the value of the angle θ in the system of cords shown in Fig. 26.

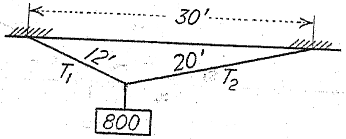


FIG. 24.

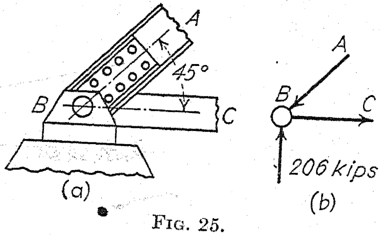


FIG. 25.

Ans. T_1 = 440 lb.; T_2 = 538 lb.; T_3 = 786 lb.; T_4 = 1205 lb.; T_5 = 1200 lb.; θ = 13°50′.

6. A boom 40 ft. long is pinned at one end and is supported by a cable passing up from the free end to a point 20 ft. vertically above the pinned end. Neglecting the weight of the boom, solve for the stress in the boom and for the stress in the cable when the boom is horizontal and carries a load of 4000 lb. at the free end. Solve for the same stresses when the cable

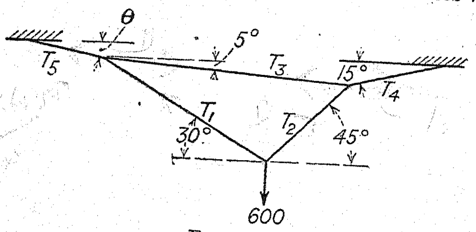


FIG. 26.

is shortened to raise the free end of the boom to the level of the cable support, so that the cable is horizontal. Prove that the stress in the boom is constant for any angle that the boom may have, either above or below the horizontal. *Ans.* 8000 lb.; 8944 lb.; 8000 lb.; 6928 lb.

20. Equilibrium of Three or More Forces: Trigonometric Solution.—In many problems the trigonometric relations between the vectors of the force polygon or the geometric relations between the vectors of the force polygon and some part of the space diagram afford easy methods of solution. In the solution of Prob. 4, Art. 19, it is seen that the force triangle is a 45°-90°-45° triangle. It is therefore evident that the same relation exists between the amounts of the three forces as exists between the lengths of the three sides of such a triangle. Stress

BC is the same amount as the vertical reaction, and stress AB is 1.414 times as much as either.

It will also be noticed that in the solution for the stresses T_1 and T_2 (Fig. 26), the force triangle had angles of 45°, 60°, and 75°. By the law of sines,

$$\frac{T_1}{\sin 45°} = \frac{T_2}{\sin 60°} = \frac{600}{\sin 75°}$$
$$\frac{T_1}{0.707} = \frac{T_2}{0.866} = \frac{600}{0.966}$$
$$T_1 = 439 \text{ lb.}; \ T_2 = 538 \text{ lb.}$$

In the same way, the values of T_3 and T_4 may be obtained. In

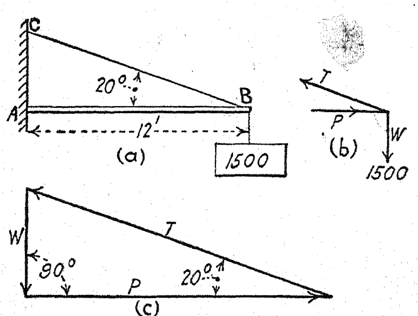


FIG. 27.

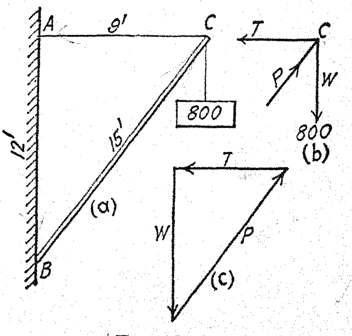


FIG. 28.

solving for T_5 in the next force triangle, it is simpler to use the cosine law $T_5^2 = T_1^2 + T_3^2 - 2T_1 \times T_3 \cos 155°$.

EXAMPLE 1

Figure 27 represents a load of 1500 lb., hung from the end of a horizontal boom AB, 12 ft. long and supported by a cable BC running up to the wall. The angle ABC is 20°. Neglect the weight of the boom, and solve for the compression P in the boom and the tension T in the cable.

Solution.—Figure 27(*b*) is the free-body diagram of point B, and Fig. 27(*c*) is the force triangle. From the trigonometry of the force triangle,

$$T = \frac{W}{\sin 20°}$$
$$T = \frac{1500}{0.342} = 4386 \text{ lb.}$$

Similarly,

$$P = \frac{W}{\tan 20°}$$
$$P = \frac{1500}{0.364} = 4120 \text{ lb.}$$

EXAMPLE 2

Figure 28(a) represents a load of 800 lb. hung from the end of a boom BC, 15 ft. long and supported by a horizontal cable AC, 9 ft. long. Solve for the stresses in the boom and the cable caused by the load.

Solution.—Figure 28(b) is the free-body diagram of point C, and Fig. 28(c) is the force triangle. The force triangle is similar to triangle ABC on the space diagram, so the corresponding sides are proportional.

$$\frac{P}{W} = \frac{15}{12}$$

$$P = 800 \times \frac{15}{12} = 1000 \text{ lb.}$$

$$\frac{T}{W} = \frac{9}{12}$$

$$T = 800 \times \frac{9}{12} = 600 \text{ lb.}$$

Problems

1. Solve for the reactions A and B of the trough on the cylinder shown in Fig. 29, assuming all surfaces to be smooth. *Ans.* 1115 lb.; 299 lb.

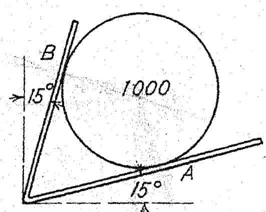


FIG. 29.

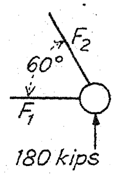


FIG. 30.

2. A weight of 3000 lb. suspended by a cable 50 ft. long is pulled 20 ft. from the vertical by a horizontal force. Solve for the amount of this horizontal force and for the resulting tension in the cable.

Ans. 1310 lb.; 3275 lb.

3. Figure 30 shows the reaction on the pin at the right end of a bridge truss, and the directions of the bottom and end chords. Solve for the stresses F_1 and F_2.

Ans. 104 kips T; 208 kips C.

4. Figure 31 represents the spreader on a drag-line bucket. Solve for the stresses in members AC and AB when the pull P is 5000 lb.

Ans. 4000 lb. T; 3125 lb. C.

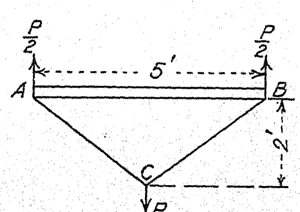


FIG. 31.

5. A boom 20 ft. long is pinned at one end and supported by a cable at the other. When the boom is at an angle of 30° above the horizontal and carries a load at the free end, what should be the angle of the supporting cable with the horizontal in order that the stress in the cable shall be a

minimum if the weight of the boom is neglected? When in this position, what is the stress in the cable and in the boom caused by a load of 2000 lb. at the free end? What are the stresses in each if the cable is horizontal?

Ans. 60°; 1732 lb.; 1000 lb.; 3464 lb.; 4000 lb.

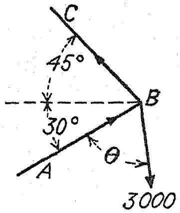


FIG. 32.

6. In Fig. 32, AB represents the end of a boom, and BC its supporting cable, both fixed in direction. The 3000-lb. force may be rotated in the vertical plane through ABC, angle θ varying from 0° to 150°. Determine the value of θ for a maximum tension in BC. Solve for the stresses in BC and AB for this position. Determine the value of θ for a maximum compression in AB, and solve for the two stresses.

Ans. 90°, 3106 lb., 804 lb.; 15°, 804 lb., 3106 lb.

21. Equilibrium of Three or More Forces: Algebraic Solution by Summation of Forces.—If for any given coplanar concurrent system of forces, $\Sigma F_x = 0$ and $\Sigma F_y = 0$, it follows that $R = 0$, and the system is in equilibrium.

Conversely: If a coplanar concurrent system of forces is in equilibrium, $R = 0$, and therefore $\Sigma F_x = 0$ and $\Sigma F_y = 0$.

Since there are only two independent equations, no more than two unknown elements can be determined. These unknown elements are usually the amounts of two of the forces, the directions being known, but they may also consist of the amount and direction of one of the forces.

The direction of the axes is immaterial. The X axis and the Y axis may be at any angle with each other.

EXAMPLE

A cast-iron sphere 1 ft. in diameter rests in an 8- by 8-in. angle, one leg of which is at an angle of 30° with the horizontal, as shown in Fig. 33(a).

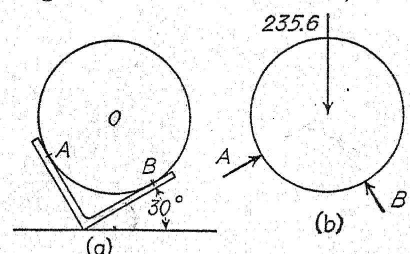


FIG. 33.

Assuming all surfaces smooth, compute the reactions on the sphere at A and B.

Solution.—The volume of a sphere $= (\frac{4}{3})\pi r^3 = 0.5236$ cu.ft. Its weight is $0.5236 \times 450 = 235.6$ lb. In Fig. 33(b), the free-body diagram is shown,

the three forces acting upon it being its weight and the reactions at A and B. The equation $\Sigma F_x = 0$ gives

$$0.866A - 0.5B = 0$$

The equation $\Sigma F_y = 0$ gives

$$0.5A + 0.866B - 235.6 = 0$$

The solution of these two equations gives $A = 117.8$ lb. and $B = 204$ lb.

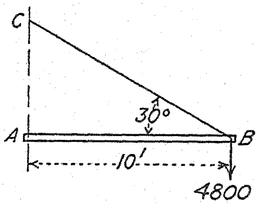


FIG. 34.

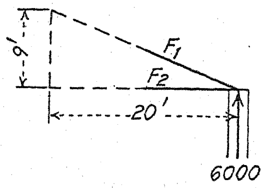


FIG. 35.

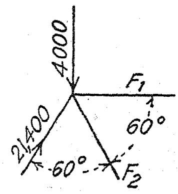


FIG. 36.

If the summation of forces is made parallel to force A, force B is eliminated, and force A is immediately determined.

$$A - 235.6 \times 0.5 = 0$$
$$A = 117.8 \text{ lb.}$$

By summing forces parallel to force B,

$$B - 235.6 \times 0.866 = 0$$
$$B = 204 \text{ lb.}$$

Problems

1. For the sphere referred to in the example above, solve for the horizontal central force necessary to reduce the reaction at B to zero. Solve also for the reaction at A. *Ans.* 408 lb.; 471 lb.

2. The boom AB, Fig. 34, is pinned at A and held by a cable BC. Solve for the stresses in AB and BC caused by the 4800-lb. load.
Ans. 8315 lb. C; 9600 lb. T.

3. Figure 35 represents one end of a roof truss. The upward reaction of the supporting wall is 6000 lb. Solve for the stresses F_1 and F_2.
Ans. 14,620 lb. C; 13,330 lb. T.

4. The system of forces shown in Fig. 36 is known to be in equilibrium. Solve for the forces F_1 and F_2. *Ans.* 19,090 lb. C; 16,780 lb. T.

5. Assuming T_1 and T_3 known (Prob. 5, Art. 19), solve for T_5 and 6 by the method of this article.

22. Equilibrium of Three or More Forces: Algebraic Solution by Moments.—If a coplanar, concurrent system of forces is in equilibrium, the resultant $R = 0$, and the moment of the resultant R with respect to any point in the plane of the force system is

zero. Since the moment of the resultant R is equal to the algebraic sum of the moments of the several forces of the system, it follows that this algebraic sum of the moments of all the forces with respect to any point in their plane is equal to zero.

If such a force system is known to be in equilibrium, and no more than two of the forces are unknown, these two may be determined. If the center of moments is chosen on the line of action of one of the unknown forces at some other than the common point of intersection, the equation $\Sigma M_o = 0$ will determine the other unknown force.

If the center of moments is not on the line of action of one of the unknown forces, the moment equation will contain two unknown quantities, and another independent equation will be needed. If another moment equation is written, the second center of moments must not be on the line through the first center of moments and the point of concurrence of the forces, since such equations are identities. For the same reason, if the second equation is obtained by a summation of forces, this summation must not be normal to the line mentioned above.

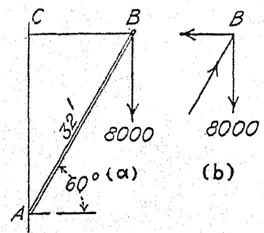


FIG. 37.

EXAMPLE

A boom AB, 32 ft. long, at an angle of 60° with the horizontal, supports a load of 8000 lb. at the end, as shown in Fig. 37(*a*). If the supporting cable CB is horizontal, determine the stresses in AB and CB.

Solution.—Point B is the free body, and Fig. 37(*b*) shows the free-body diagram. Distance CB is $32 \times 0.5 = 16$ ft., and distance AC is $32 \times 0.866 = 27.71$ ft.

The equation $\Sigma M_A = 0$ gives

$$CB \times 27.71 = 8000 \times 16$$
$$CB = 4620 \text{ lb.}$$

The equation $\Sigma M_C = 0$ gives

$$AB \times 16 \times 0.866 = 8000 \times 16$$
$$AB = 9240 \text{ lb.}$$

Problems

1. Solve the example above if the force of 8000 lb. is acting at an angle of 60° with AB in the vertical plane.

Ans. $CB = 8000$ lb. T; $AB = 8000$ lb. C.

2. Solve the foregoing example if point C is moved 6 ft. downward and CB remains unchanged in length, thus increasing the angle of the boom with the horizontal. *Ans.* CB = 5900 lb. T; AB = 11,800 lb. C.

3. Figure 38 represents the first panel at the left end of a bridge truss. Compute the stresses F_1 and F_2. *Ans.* 32,000 lb. T; 51,200 lb. C.

4. A boom 48 ft. long supports a load of 2000 lb. as shown in Fig. 39. Compute the stresses in AC and BC when the load hangs vertically.

Ans. 5330 lb. C; 5700 lb. T.

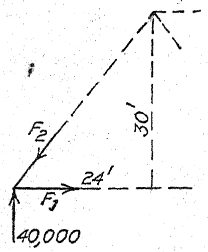


FIG. 38.

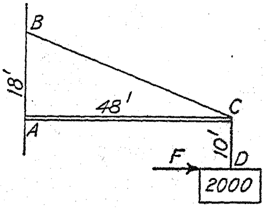


FIG. 39.

5. What horizontal force F applied to the weight D as shown in Fig. 39 will push it 3 ft. from the vertical? Compute the stresses in CD and AC. Prove without computing it that the stress in BC will remain unchanged.

Ans. 630 lb.; 2095 lb.; 4700 lb.

GENERAL PROBLEMS ON COPLANAR, CONCURRENT FORCES

1. A weight of 1000 lb. is supported by two cords, one at an angle of 15°, the other at an angle of 5°, with the horizontal. Solve for the tensions in the two cords. *Ans.* 2910 lb.; 2830 lb.

2. The left end of a wire for a slack-wire performer is 80 ft. horizontally from the right end and 15 ft. below it. When the performer, weighing 140 lb., is 20 ft. horizontally from the left end, that point is 3 ft. below the level of the left end. Assuming the wire to be flexible and neglecting its weight, solve for the stresses in the two ends of the wire.

Ans. Left end, 315 lb.; right end, 325 lb.

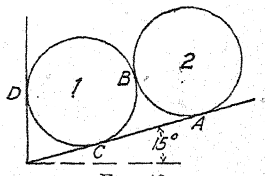


FIG. 40.

3. The cylinders shown in Fig. 40 are the same diameter, but cylinder 1 weighs 800 lb. and cylinder 2 weighs 1200 lb. Solve for the reactions A, B, C, and D, assuming all surfaces smooth.

Ans. A = 1159 lb.; B = 311 lb.; C = 911 lb.; D = 536 lb.

4. In Fig. 41, cylinders 1 and 3 are each 2 ft. in diameter and each weighs 400 lb. Cylinder 2 is 3 ft. in diameter and weighs 900 lb. Assuming all surfaces smooth, solve for the reactions A, B, C, D, E, and F.

Ans. A = 387 lb.; B = 204 lb.; C = 606 lb.; D = 664 lb.; E = 970 lb.; F = 982 lb.

5. In Fig. 41, what horizontal central force acting toward the left on cylinder 3 will reduce the reaction at A to zero? Solve also for the reactions B, C, D, E, and F.

Ans. 1195 lb.; B = 1260 lb.; C = 183 lb.; D = 1720 lb.; E = 1780 lb.; F = 2180 lb.

6. A picture weighing 80 lb. is hung by an endless wire passing through the four screw rings in the back and over a hook at A, as shown in Fig. 42.

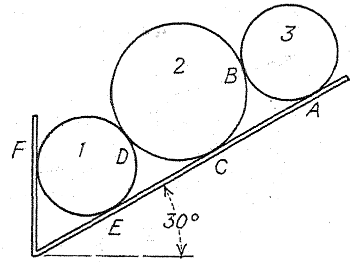


FIG. 41.

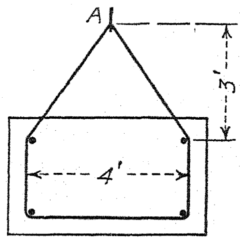


FIG. 42.

Solve for the tension in the wire and for the amount and direction of the resultant pull on each ring.

Ans. 48 lb.; upper rings, 27.8 lb., 16°40′ with X; lower rings, 67.9 lb., 45° with X.

7. Solve for the reactions A, B, C, and D on the two cylinders shown in Fig. 43, assuming all surfaces to be smooth.

Ans. A = 124 lb.; B = 159 lb.; C = 400 lb.; D = 124 lb.

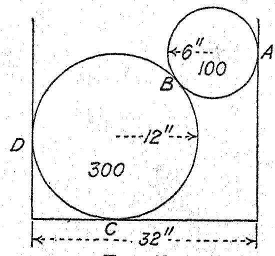


FIG. 43.

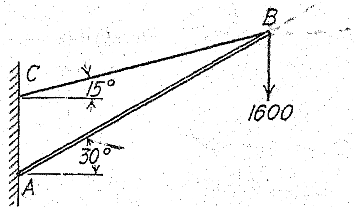


FIG. 44.

8. A weight of 10,000 lb. is supported by two cables. One is at an angle of 30° with the horizontal, and the stress in it cannot exceed 15,000 lb. What is the minimum angle with the horizontal that the other cable may have, and what will be the stress in it? *Ans.* 10°55′; 13,240 lb.

9. If the stress in each cable in Prob. 8 is not to exceed 12,000 lb., what is the minimum angle that each cable may have with the horizontal?

Ans. 24°40′.

10. In Fig. 44, the boom AB is pinned to the wall at A. Solve for the stresses AB and BC due to the 1600-lb. load. *Ans.* 5970 lb.; 5350 lb.

11. Figure 45 represents a simple derrick. Solve for the stresses in AB and AC. *Ans.* AB = 4500 lb. T; AC = 4500 lb. C.

12. Assuming that the 3000-lb. pull may be rotated in the vertical plane through AC, solve for the maximum stress that may be caused in AB and for the corresponding stress in AC, Fig. 45.

Ans. AB = 4770 lb. T; AC = 3710 lb. C.

13. A wheel 2 ft. in diameter carries a load of 1600 lb., as shown in Fig. 46. Solve for the horizontal force P applied at the center which is necessary to

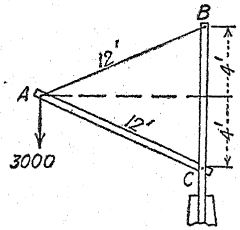


FIG. 45.

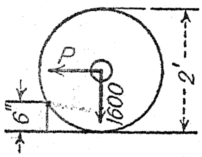


FIG. 46.

start it over the 6-in. block and for the reaction at the block. Solve also for the minimum force P to start the wheel over the block, the angle it makes with the horizontal, and the reaction at the block.

Ans. 2771 lb.; 3200 lb.; 1386 lb.; 60°; 800 lb.

14. Solve Prob. 13 if the wheel is 6 ft. in diameter.

Ans. 1061 lb.; 1920 lb.; 884 lb.; 33°30′; 1333 lb.

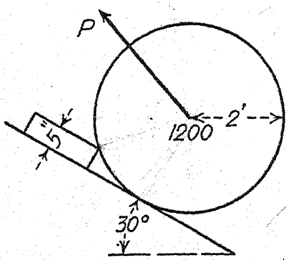


FIG. 47.

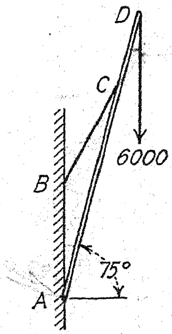


FIG. 48.

15. Solve for the amount and direction of the least pull P to start the wheel shown in Fig. 47 over the block. Solve also for the reaction at the block.

Ans. 1110 lb.; 67°40′; 456 lb.

16. In the crane shown in Fig. 48, AB = 30 ft., AD = 80 ft., and CD = 20 ft. Solve for the stress in the tie rod BC and for the amount and direction of the hinge reaction at A due to the 6000-lb. load.

Ans. 8530 lb.; 14,060 lb.; 72°50′.

CHAPTER III

COPLANAR, PARALLEL FORCES

23. Space Diagrams and Force Diagrams. *Bow's Notation.*—As explained in Art. 11, it is more convenient to use separate space and force diagrams in the graphical work for all except the simplest problems. A system of lettering of these two diagrams, known as *Bow's notation*, is very useful in checking results and will be used in such problems in this book.

In the space diagram, each space from the line of action of one force to that of the next one is lettered with a lower-case letter,

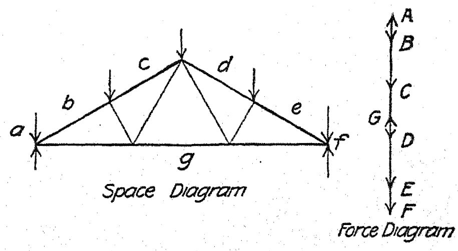


Fig. 49.

and the line of action of any force is designated by the letters of the two spaces that it separates, read clockwise around the diagram. The line of action *bc* in Fig. 49 is the line between space *b* and space *c*. The corresponding upper-case letters are placed at the ends of the corresponding vector in the force diagram. Thus vector *BC* represents in amount the force acting along line *bc* in the space diagram, and the direction is from *B* to *C*, downward. Vector *GA* represents the amount of the reaction of the support acting along line *ga*, and the direction is from *G* to *A*, upward.

24. Resultant of Two Parallel Forces, Graphically.—In finding the resultant of two parallel forces, the same method may be used as that in Art. 12, the case in which the two forces, while concurrent, do not intersect on the diagram. In Fig. 50, let vectors *AB* and *CD* represent two parallel forces which are to be com-

bined into their resultant. The system is not changed by adding to it the two equal, opposite, collinear forces AE and CF. Vectors AB and AE are combined into their resultant AG, and vectors CD and CF into their resultant CH. The two resulting forces of the system, AG and CH, are concurrent at M and are transmitted along their lines of action to positions ML and MK. By the parallelogram law, they are then combined into their resultant MN, which is therefore the resultant of the original system. Vector MN is equal to the sum of the original vectors AB and CD.

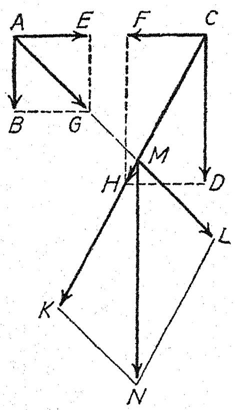


FIG. 50.

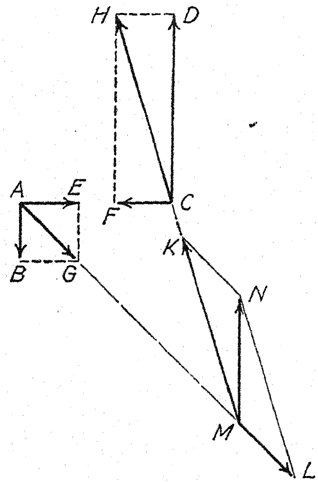


FIG. 51.

The case in which the forces are opposite in direction and unequal in amount is shown in Fig. 51, and the discussion used for Fig. 50 applies also to Fig. 51. Vector MN is equal to the *algebraic* sum of the original vectors AB and CD.

If the two oppositely directed forces are equal, they form a *couple*, as will be discussed in Art. 31. They cannot be combined into a single force.

Problems

1. In Fig. 50, let AB = 60 lb., CD = 140 lb., and the distance AC = 6 ft. Solve for the amount of the resultant and its distance from the line of action of AB. *Ans.* 200 lb.; 4.2 ft.

2. Solve Prob. 1 if the data given there apply to Fig. 51. *Ans.* 80 lb.; 10.5 ft.

25. Principle of Moments for Two Coplanar, Parallel Forces.—By the principle of moments for two concurrent forces (Art. 18), the algebraic sum of the moments of forces AB and CD in either Fig. 50 or Fig. 51 with respect to any point in their plane is equal to the sum of the moments of forces AG and CH with respect to the same point, since the moment of the added forces AE and CF is zero. By the same principle, the sum of the moments of forces AG and CH (ML and MK) with respect to any point in their plane is equal to the moment of their resultant MN with respect to the same point. The *principle of moments* for two parallel forces may now be stated:

The algebraic sum of the moments of two parallel forces with respect to any point in their plane is equal to the moment of their resultant with respect to the same point.

In Fig. 52, let MN be the resultant of forces AB and CD. Since the moment of MN with respect to any center on its line of action, such as O, is zero, it follows from the equation of moments that the algebraic

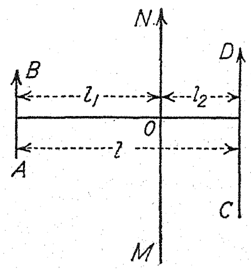


FIG. 52.

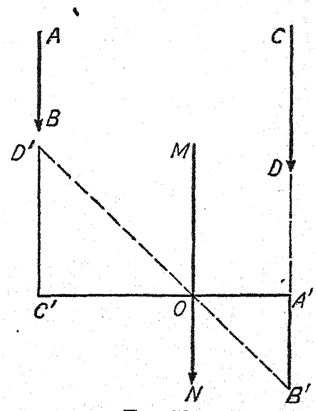


FIG. 53.

sum of the moments of the two components AB and CD with respect to the same axis must be zero.

$$AB \times l_1 - CD \times l_2 = 0$$

$$\frac{AB}{CD} = \frac{l_2}{l_1}$$

The principle embodied in the equation just derived may be stated as follows: The perpendicular distances of the resultant from the forces are to each other inversely as the forces.

Since by geometry three parallel lines divide any two intersecting straight lines in the same ratio, this principle holds true for any diagonal distances, also.

The graphical application of this principle is often very advantageous, so will be given here. If AB and CD, Fig. 53, are the two forces, any convenient straight line, such as $C'A'$, may be drawn between their lines of action. On the line of action of AB, length $C'D'$ is laid off to represent CD to some scale. On the line of action of CD, length $A'B'$ to represent AB to the same scale is laid off in the opposite direction. The diagonal line $B'D'$ intersects $C'A'$ at O, the point through which the resultant MN must act.

In case the two forces are opposite in direction, their vectors must be laid off in the same direction from the base line instead of in opposite directions. The intersection of the base line and the line joining the ends of these vectors will be outside the two forces, on the side of the larger force.

This method of combining forces graphically is called the method of *inverse proportion.*

The inverse of the problem just solved is that of breaking up a given force into two parallel components acting through given points. Let AB, Fig. 54, be the force that is to be broken up into parallel components at M and N. Draw lines CM and DN parallel to AB. Through point A draw line AC to any convenient point on CM, and through point B draw line BD parallel to AC to intersect ND. Join points C and D by a straight line which intersects vector AB at O. Vector AO represents the component acting at N, and vector OB represents the component acting at M.

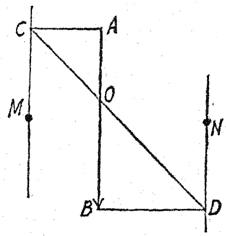


FIG. 54.

Problems

1. In Fig. 52, let AB = 8000 lb., CD = 10,000 lb., and length l = 25 ft. Solve for the resultant and the distance l_1. *Ans.* 18,000 lb.; 13.89 ft.

2. Solve Prob. 2, Art. 24, by inverse proportion.

26. Resultant of Two Parallel Forces, Algebraically.—The resultant of two parallel forces is given in amount and direction by the algebraic sum of the two components. The principle of moments locates the line of action of the resultant with respect to any chosen point of reference. If two parallel forces are equal and act in the same direction, their resultant acts midway between them.

Problems

1. Forces of 600 and 2200 lb., respectively, are acting in the same direction 20 ft. apart. Locate the resultant with respect to the 600-lb. force.
Ans. 15.71 ft.

2. Solve Prob. 1 if the forces act in opposite directions. *Ans.* 27.5 ft.

27. Resultant of Three or More Parallel Forces, Graphically.—The graphical method of Art. 24 is readily extended to the case of three or more parallel forces in a plane. When there are more than two forces, it is more convenient to use both space and force diagrams, as shown in Fig. 55. Three forces are shown, AB, BC, and CD, laid down in order in the force diagram and acting

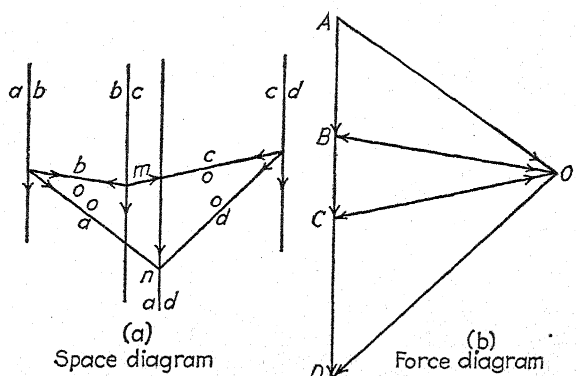


FIG. 55.

along lines ab, bc, and cd, respectively, in the space diagram. In the force diagram, any two convenient forces BO and OB equal, opposite, and collinear are added to the system, acting along the line bo in the space diagram through point m. The addition of these two forces to the system does not change its effect in any way. Forces AB and BO, acting along ab and bo, are now combined into force AO, acting along ao in the space diagram. The two equal, opposite, collinear forces CO and OC are now added, also acting through point m along line co. Force CO now balances the two forces OB and BC, so these three are eliminated. Next, forces OC and CD, acting along oc and cd, are combined into their resultant OD, acting along od. The system is now reduced to the two concurrent forces AO and OD, acting along ao and od and meeting at n. These are now combined into their

resultant AD, acting along ad in the space diagram. Force AD, acting through n, is therefore the resultant of the original system.

In a similar manner, any number of additional forces may be added to the system. Forces may be acting in either direction.

The lines AO, BO, etc., in the force diagram are called *rays*, and the lines ao, bo, etc., in the space diagram are called *strings*. The polygon that the strings make is called the *funicular* polygon. In making the graphic solution, it is not necessary to place arrowheads on the rays and strings. The steps of the solution in order may be outlined as follows:

1. Draw the space diagram.
2. Draw the force polygon, noting the resultant.
3. Choose any convenient pole O, and draw the rays.
4. Parallel to the rays of the force diagram, draw the corresponding strings of the funicular polygon in the space diagram.
5. The intersection of the first and last strings determines the position of the resultant.

If Bow's notation is used, each string has its corresponding ray lettered similarly. For example, string ob is parallel to ray OB and is drawn between the two lines that enclose the b space, ab and bc. When the solution is begun, string oa is "free" until intersected by the other free string to determine the line of action of the resultant.

If the final point of the force polygon coincides with the initial point so that the resultant $R = 0$ but the two "free" strings do not coincide, the resultant of the system is a couple, as will be discussed in Art. 31.

The method of inverse proportion explained in Art. 25 may also be extended to apply to the case of three or more parallel forces. Two of the forces may be combined into their resultant; then this resultant may be combined with a third; and so on until all the forces are combined.

EXAMPLE

Combine into their resultant the five forces shown in Fig. 56.

Solution.—The space diagram, Fig. 57(a), is drawn first, showing the lines of action of the forces. The lines of action of the two forces below the truss are extended upward, so that all five lines are lettered above the truss. In Fig. 57 (b), the five vectors are drawn to scale, from A to F. Vector AF, 12 kips to scale and acting from A to F, is then the resultant. Point O is selected, and rays OA, OB, etc., are drawn. Beginning at point m on

ab, the free string *ao* is drawn, parallel to *AO*; then strings *ob*, *oc*, *od*, and *oe*, parallel, respectively, to the corresponding rays in the force diagram; and finally the free string *of*, parallel to *OF*. The intersection of these free strings *ao* and *of* at *n* determines the line of action *af* of the resultant force *AF*. The distance from the left end of the truss scales 16.67 ft.

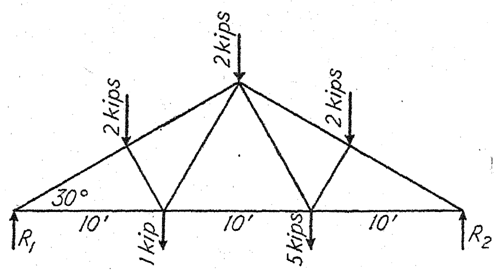


FIG. 56.

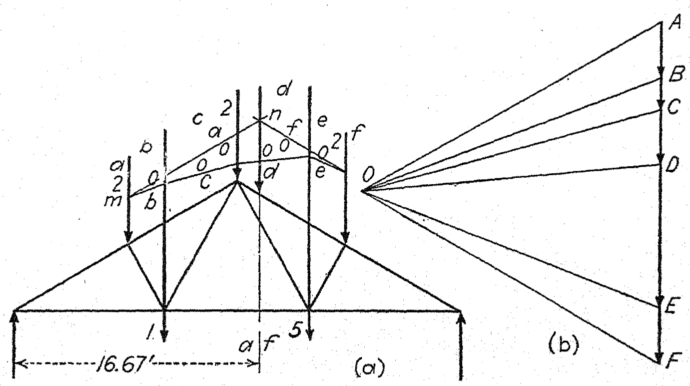


FIG. 57.

Problems

1. The downward vertical loads on a 30-ft. horizontal beam are as follows: 400 lb. 5 ft. from the left end; 900 lb. at the middle; 1200 lb. at the right end. Determine the amount and position of the resultant.

Ans. 2500 lb., 20.6 ft. from left end.

2. Solve the example above if the force of 1 kip acts upward on the truss instead of downward. *Ans.* 10 kips, 18 ft. from left end.

28. Resultant of Three or More Parallel Forces, Algebraically. It is evident that the amount and direction of the resultant *R* of a system of three or more parallel forces will be given by their algebraic sum. In order to determine the position of the resultant, the principle of moments, stated in Art. 25, is extended to the case of three or more forces. It was shown there that the

algebraic sum of the moments of two parallel forces with respect to any point in their plane is equal to the moment of their resultant with respect to the same point. Any two of the forces may be combined into their resultant; this resultant in turn may be combined with another force; and so on until all the forces are combined into the single resultant of the whole system.

The algebraic sum of the moments of any number of coplanar parallel forces with respect to any point in their plane is equal to the moment of their resultant with respect to the same point.

If a is the perpendicular distance of the resultant R from any point of reference, and a_1, a_2, etc., the perpendicular distances of F_1, F_2, etc., respectively, from the same point of reference,

$$F_1a_1 + F_2a_2 +, \text{ etc.} = Ra$$

$$a = \frac{\Sigma Fa}{R}$$

EXAMPLE

Determine the amount, direction, and position of the resultant of the four forces shown in Fig. 58.

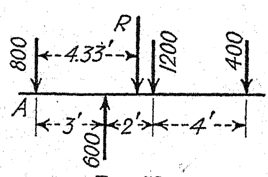


FIG. 58.

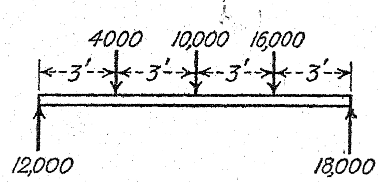


FIG. 59.

Solution.—The summation of forces gives $800 + 1200 + 400 - 600 = 1800$ lb. downward. Let a be the distance from point A to the line of action of the resultant. Then

$$1800 \times a = (1200 \times 5) + (400 \times 9) - (600 \times 3)$$

$$a = 4.33 \text{ ft.}$$

Problems

1. Determine the position of the resultant of the three downward forces shown in Fig. 59. *Ans.* 7.2 ft. from left end.

2. Determine the position of the resultant of the two upward forces shown in Fig. 59.

3. Reverse the direction of the 400-lb. force in Fig. 58, and solve for the amount, direction, and position of the resultant.

Ans. 1000 lb. downward, 0.6 ft. to the right of A.

29. Equilibrium of Parallel Forces: Graphic Solution.—If in the graphic solution explained in the first part of Art. 27 the

final point of the force polygon coincides with the initial point, the resultant R of the system is zero. If also, in the space diagram, the two free strings coincide, the resultant moment is zero, and the force system is in equilibrium.

Conversely: If a system of coplanar parallel forces is in equilibrium, the force polygon must close, and the funicular polygon must close.

By the application of these two conditions of equilibrium, two unknown forces may be determined.

The method described in the last paragraph in Art. 27 may also be extended to apply to the determination of two unknown forces in a system of parallel forces known to be in equilibrium. By the method of inverse proportion, the known forces may all be combined into a single resultant. Since the original system was in equilibrium, their resultant and the two unknown forces must be in equilibrium. If, now, by the method of Art. 25 their resultant is resolved into two parallel components acting along the same lines of action as the two unknown forces, the new system of four forces must be in equilibrium. Each one of the two unknown forces must be equal and opposite to the components just obtained and so are completely determined.

EXAMPLE 1

Determine the reactions R_1 and R_2 of the beam shown in Fig. 60.

Solution.—The space diagram, Fig. 60(a), is drawn to scale. The force polygon, Fig. 60(b), is drawn to scale as far as known, AB, BC, CD. Since

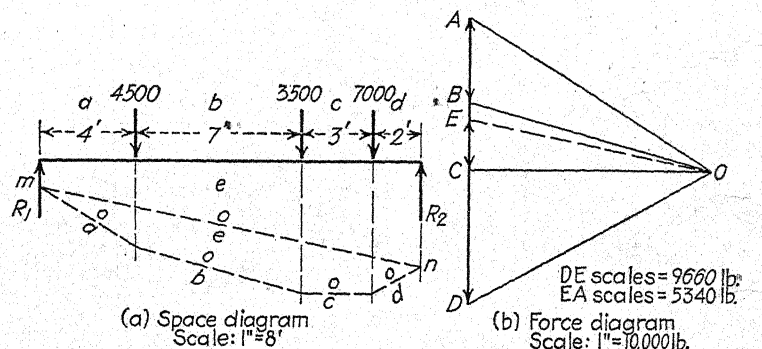


FIG. 60.

the system is in equilibrium, point A must be the closing point, but point E is unknown. The location of point E is determined by the fact that the funicular polygon must close also. Any convenient pole O is selected, and

the rays OA, OB, OC, and OD are drawn. The funicular polygon is begun at any convenient point on any one of the forces, as point m on force R_1. String oa is drawn parallel to ray OA across the a space. From the point at which oa intersects ab, ob is drawn parallel to OB across the b space. Strings oc and od are drawn in a similar way. Since the funicular polygon must close, string oe must necessarily run from m to n. In the force diagram, ray OE must be parallel to string oe in the space diagram, so point E is determined. Vector DE represents the reaction R_2 to scale, and vector EA represents the reaction R_1.

EXAMPLE 2

Determine the reactions R_1 and R_2 of the beam shown in Fig. 61.

Solution.—In Fig. 61(b), AB is laid off 8000 lb. to scale on the line of action of the 2000-lb. force, and CD is laid off in the opposite direction

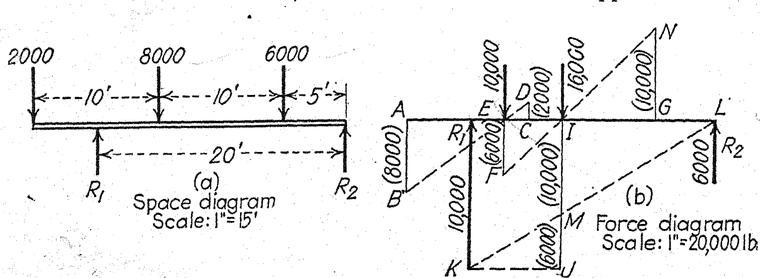


FIG. 61.

from base AL 2000 lb. to scale on the line of action of the 8000-lb. force. The line BD crosses the base line at E and locates the position of the 10,000-lb. resultant. In a similar manner, point I on the resultant of this 10,000-lb. force and the remaining 6000-lb. force is located. IJ is laid off 16,000 lb. to scale. JK is drawn horizontally to intersect reaction R_1, and IL similarly to intersect R_2. Line KL intersecting IJ at M divides IJ into IM = 10,000 lb., and MJ = 6000 lb. Component IM acts along R_1, and component MJ acts along R_2. Reaction R_1 must therefore be 10,000 lb. upward, and reaction R_2 must be 6000 lb. upward.

Problems

1. Solve Example 1 if an additional downward force of 5000 lb. is added at a point 2 ft. from the left end.

Ans. 10,280 lb.; 9720 lb.

2. A uniform beam 20 ft. long, supported at the ends, weighs 3600 lb. and carries a load of 6400 lb. at a point 5 ft. from the left end and one of 8000 lb. at a point 4 ft. from the right end. Solve for the two reactions.

Ans. 8200 lb.; 9800 lb.

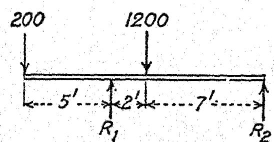


FIG. 62.

3. Solve for the reactions of the overhanging beam shown in Fig. 62.

Ans. 1244 lb.; 156 lb.

30. Equilibrium of Parallel Forces: Algebraic Solution.—If for any system of coplanar parallel forces $\Sigma F = 0$, there is no resultant force, and the system is in equilibrium in translation. If also $\Sigma M = 0$ with respect to any axis normal to the plane of the forces, there is no resultant moment, and the system is in equilibrium in rotation.

Conversely: If a system of coplanar parallel forces is in equilibrium, the resultant $R = 0$, and the moment M with respect to any axis $= 0$.

If, in a given system of forces known to be in equilibrium, some of the forces are unknown, they may be determined by applying the conditions of equilibrium. The number of unknown forces, however, must not exceed the number of *independent* equations that can be written, two in this case.

If some point on the line of action of one of the unknown forces of a system is used as the center of moments, there is only one unknown quantity in the equation, so this unknown quantity is immediately determined. The second equation may then be another moment equation or the equation of the summation of forces.

In solving for the unknown reactions of a simply supported beam, the method of inverse proportion is often advantageous. If the loading and supports of a simply supported beam are symmetrical, no equations are necessary, since each reaction is one-half the total load.

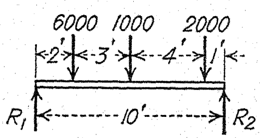


FIG. 63.

EXAMPLE

A beam 10 ft. long, supported at the ends, carries three loads spaced as shown in Fig. 63. Solve for the reactions R_1 and R_2.

Solution.—Equation $\Sigma M = 0$, with any point on R_2 as the center of moments, gives

$$10R_1 - (2000 \times 1) - (1000 \times 5) - (6000 \times 8) = 0$$
$$R_1 = 5500 \text{ lb.}$$

With any point on R_1 as the center of moments, the moment equation gives

$$10R_2 - (6000 \times 2) - (1000 \times 5) - (2000 \times 9) = 0$$
$$R_2 = 3500 \text{ lb.}$$

Instead of the second moment equation, the equation $\Sigma F = 0$ could have been used.

$$R_2 + 5500 = 9000$$
$$R_2 = 3500 \text{ lb.}$$

The use of both equations gives a check on the correctness of the solution.

By the method of inverse proportion,

$$R_1 = \tfrac{8}{10} \times 6000 + \tfrac{5}{10} \times 1000 + \tfrac{1}{10} \times 2000 = 5500 \text{ lb.}$$
$$R_2 = \tfrac{9}{10} \times 2000 + \tfrac{5}{10} \times 1000 + \tfrac{2}{10} \times 6000 = 3500 \text{ lb.}$$

It will be noticed that these are the moment equations with the moment arms of R_1 and R_2 divided out.

Problems

1. Solve for the reactions R_1 and R_2 of the beam supported and loaded as shown in Fig. 64. *Ans.* 5667 lb.; 4333 lb.

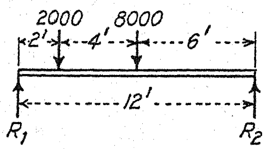


FIG. 64.

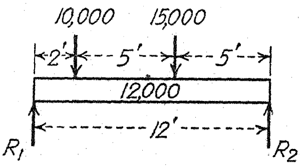


FIG. 65.

2. Solve for the reactions R_1 and R_2 of the beam shown in Fig. 65. The weight of the beam is uniformly distributed. *Ans.* 20,580 lb.; 16,420 lb.

3. Solve for the reactions R_1 and R_2 of the overhanging beam shown in Fig. 66. *Ans.* −~~550 lb.; 4550~~ lb.

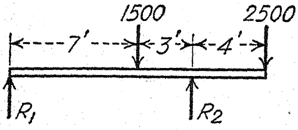


FIG. 66.

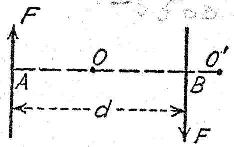


FIG. 67.

31. Couples.—Two parallel forces, equal in amount, opposite in direction, and with different lines of action, constitute a couple, as F,F, Fig. 67. The perpendicular distance between them d is called the *arm* of the couple. The product of one force and the arm is called the *moment* of the couple, or

$$\text{Moment} = Fd.$$

The moment of a couple is the same with respect to any point in its plane, as will be shown. Let O and O' be any two points in the plane of the couple, Fig. 67.

$$\Sigma M_o = F \times \overline{OA} + F \times \overline{OB} = F \times \overline{AB} = Fd.$$

Also

$$\Sigma M_o' = F \times \overline{O'A} - F \times \overline{O'B} = F \times \overline{AB} = Fd.$$

Since the resultant R of a couple is zero, moment is the only effect of a couple. It follows, then, from the two preceding principles, that a couple may be transferred to any place in its plane or rotated through any angle in its plane and its effect will remain the same.

Since moment is the only effect of a couple, it follows also that any couple may be replaced by another of the same moment in the same plane. Thus the rotary effect of a couple composed of two forces of 8 pounds each, acting 3 feet apart, is the same as that of another in the same direction with forces of 4 pounds each, acting 6 feet part.

No single force can balance a couple. Since the resultant R of the couple is zero, the resultant of the couple and another force could not be zero.

A couple may be transferred to any plane parallel to its original plane without change of effect. Since the moment of a couple with respect to any point O in its plane is the same as its moment with respect to an axis through O, perpendicular to its plane, the moment is independent of the location of the plane of the couple along the axis. For example, if a steam pipe is being screwed into a sleeve by means of two pipe wrenches (constituting a couple), the effect is the same, no matter at what point along the pipe they are applied.

Since couples have no properties but *magnitude* and *direction*, they may be represented graphically by vectors. The length of the vector represents to some scale the magnitude of the couple, and the direction of the vector shows the direction of its plane and the direction of its rotation. The vector is drawn perpendicular to the plane of the couple. The convention commonly used with regard to the arrow is that in which, if the couple is viewed from the head end of the vector, the rotation of the couple appears positive (counterclockwise).

The position of the vector is immaterial, since the moment of the couple is the same with respect to any axis perpendicular to its plane.

The moment of the resultant of any number of coplanar couples or of couples in parallel planes is equal to the algebraic sum of the moments of the component couples.

Couples may be compounded by combining their vectors. Since the position of the vector is immaterial, the vectors of the

couples may all be taken through any given point, then added graphically. The resultant vector represents completely the resultant couple.

Problems

1. Compute the moment of the two forces shown in Fig. 68 with respect to point C. *Ans.* +12,470 lb.-ft.

2. Figure 69 represents a horizontal plate, in the plane of which the three couples are acting. If a scale of 1 in. = 100 lb.-in. is used in representing the couples graphically by vectors, what must be the length and direction of the vector representing the resultant of the three couples? *Ans.* 2.6 in., upward.

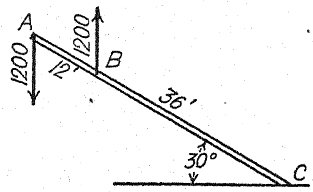


FIG. 68.

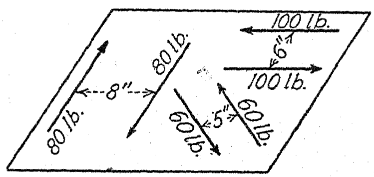


FIG. 69.

32. Resolution of a Force into a Force at a Given Point and a Couple.—In the solution of some problems, it is convenient to replace a given force by a force through some other point and a couple. In Fig. 70, F represents any given force acting through point A, and O is any chosen point. At point 0, the two opposite, collinear forces F_1 and F_2, each equal in amount to force F and parallel to it, may be added to the system with no change of effect. Forces F and F_1 now constitute a couple, the moment of which is Fd. This couple may be transferred anywhere in its plane or into any parallel plane, leaving the force F_2, equal to F in amount and having the same direction but acting at point O.

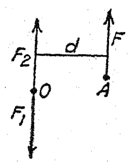


FIG. 70.

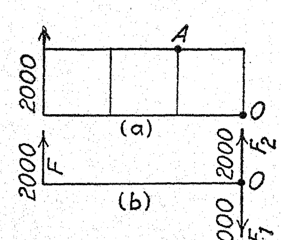


FIG. 71.

EXAMPLE

Resolve the force shown in Fig. 71(a) into a force at O and a couple. The rectangle is 3 ft. wide and 1 ft. high.

Solution.—Through point O, Fig. 71(b), apply additional forces, F_2 equal and parallel to F and in the same direction, and F_1 equal and parallel to F and in the opposite direction. The system of three forces is now exactly equivalent to the original force, since F_1 and F_2 neutralize each other.

If, now, F_1 is paired with F, the two form a couple that has a moment of -6000 lb.-ft. There remains force F_2, equal to F and in the same direction but acting at O. It should be noticed that the force is the same in amount and direction as the original force and that the moment of the couple is the same in amount and direction as the moment of the original force about point O.

Problems

1. Resolve the 2000-lb. force in Fig. 71 into a force at A and a couple. If the couple is transferred in its plane and its forces are made to act through points O and A, what is the minimum amount that the forces may have?

Ans. -4000 lb.-ft.; 2828 lb.

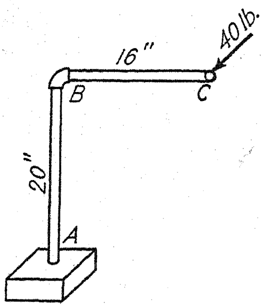


FIG. 72.

2. Figure 72 represents a piece of gas pipe ABC which is held at A and twisted by means of a pressure of 40 lb. normal to the pipe in the horizontal plane at point C. Resolve the force at C into a force at B and a couple in order to determine the twisting effect on AB and the bending effect at point A.

Ans. 640 lb.-in.; 800 lb.-in.

GENERAL PROBLEMS ON COPLANAR, PARALLEL FORCES

1. Solve for the amount and position of the resultant of the loads on the cantilever truss shown in Fig. 73. *Ans.* 15 kips, 7.6 ft. from AB.

2. In Fig. 74, B is the middle point of AE, and BC is normal to AB. Solve for the amount and position of the resultant of the three loads.

Ans. 12 kips, 12.65 ft. from A.

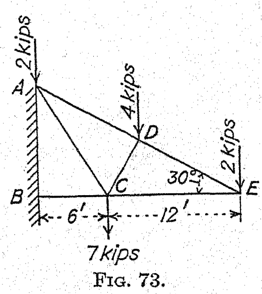


FIG. 73.

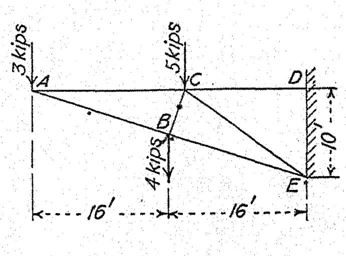


FIG. 74.

3. Solve for the amount and position of the resultant of the loads acting on the truss shown in Fig. 75. Solve also for the reactions R_1 and R_2.

Ans. 23 kips, 19.3 ft. from R_1; 10.67 kips; 12.33 kips.

4. Solve for the amount of the resultant of the five forces on the truss shown in Fig. 76. Locate the point at which it intersects the lower chord BC.

Ans. 18 kips; 12.2 ft.

5. Solve for the reactions R_1 and R_2 of the beam shown in Fig. 77. *Ans.* 4625 lb.; 3375 lb.

6. Solve for the reactions R_1 and R_2 of the overhanging beam shown in Fig. 78. *Ans.* 22,750 lb.; 31,250 lb.

7. Solve for the reactions R_1 and R_2 of the beam shown in Fig. 79. What additional load may be placed at the left end before reducing reaction R_2 to zero? *Ans.* 26,840 lb.; 4960 lb.; 10,540 lb.

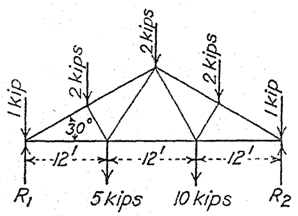


FIG. 75.

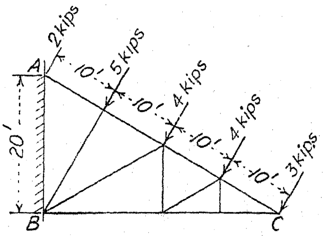


FIG. 76.

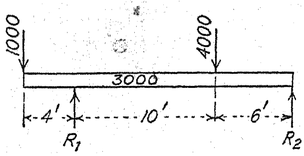


FIG. 77.

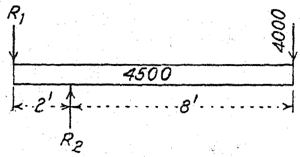


FIG. 78.

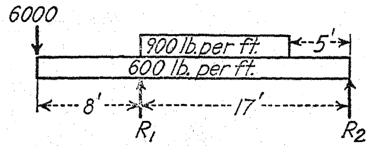


FIG. 79.

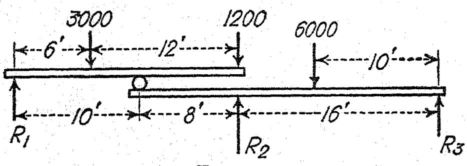


FIG. 80.

8. The sliding counterpoise weight of a steelyard weighs 4 lb. The length of the short arm is 2 in. How far apart must the pound graduations be placed on the beam? *Ans.* ½ in.

9. Solve for the reactions R_1, R_2, and R_3 of the beam system shown in Fig. 80 due to the concentrated loads. *Ans.* 240 lb.; 9690 lb.; 270 lb.

10. Figure 81 represents a three-horse evener. With the dimensions shown, what percentage of the total tractive effort P does each of the horses A, B, and C exert? *Ans.* 34.3; 30.9; 34.8.

11. If with the evener shown in Fig. 81 horse B lags until he pulls nothing, what percentage of the tractive effort do horses A and C each exert?
Ans. 46.9; 53.1.

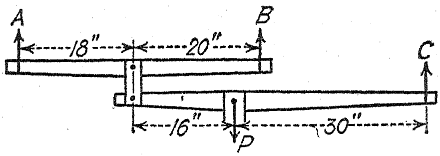


FIG. 81.

12. A rubber-tired farm tractor weighs 2500 lb.; its wheel base is 80 in.; and its center of gravity is 30 in. in front of the drive wheels. If the point of attachment of its load is 20 in. above the ground, and the coefficient of friction between the tires and the ground is 0.4, what is the maximum tractive effort that can be exerted? What are the vertical reactions on the front wheels and on the rear wheels? What are these reactions when the tractor is not exerting any tractive effort?
Ans. 695 lb.; 765 lb.; 1735 lb.; 940 lb.; 1560 lb.

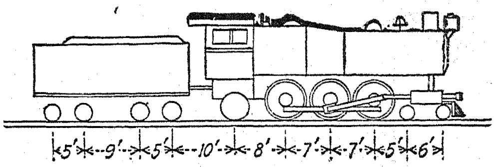


FIG. 82.

13. The sketch in Fig. 82 represents an A. T. & S. F. passenger locomotive, with dimensions given to the nearest foot. The weight on the driving wheels is 147,400 lb.; that on the front truck is 28,600 lb.; and that on the trailers is 38,600 lb. The weight of the tender is 128,000 lb. Solve for the distance of the front truck wheels from the edge of a turntable 80 ft. in diameter for perfect balance.
Ans. 8.7 ft.

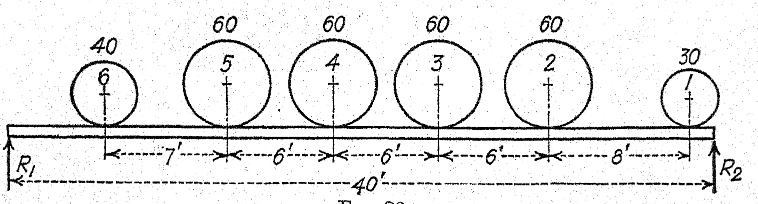


FIG. 83.

14. As a series of wheel loads such as those shown in Fig. 83 move across a beam, the maximum moment under any wheel is produced when that wheel is as far one way from the middle of the beam as the center of gravity of the entire system is the other way. Locate the position of wheel 4 from

the left end of the beam for a maximum moment under it. Solve for the reactions R_1 and R_2 when in this position. The loads are given in kips. *Ans.* 18.7 ft.; 144.9 kips; 165.1 kips.

15. Solve Prob. 14 if there are no trailer wheels (No. 6) and the loads on the other wheels remain the same. *Ans.* 17.56 ft.; 118.5 kips; 151.5 kips.

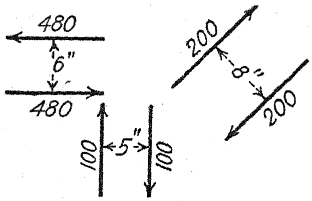


FIG. 84.

16. Combine the three couples shown in Fig. 84 into a single resultant couple. If the forces of a balancing couple are to be 4 in. apart, what must be the amount of each force? *Ans.* 780 lb.-in.; 195 lb.

17. A workman closes a gate valve by exerting a pressure of 30 lb. with each hand at opposite sides on the rim of a hand wheel 2 ft. in diameter. At another time he thrusts a bar through the wheel and exerts a pressure at only one side, 40 in. out from the center. What pressure must he exert? What is the difference in the action in the two cases? *Ans.* 18 lb.

CHAPTER IV

COPLANAR, NONCONCURRENT FORCES

33. Two-force and Multiple-force Members.—In the analysis of a common roof or bridge truss, the assumption is commonly made that the stress acts axially along each member. The trusses are assumed to be "pin-connected"; that is, each member is assumed to extend only from one joint to the next and to be connected to the other members by pins. The weight of the member itself is sometimes negligible compared with the other forces acting upon it. If the weight is not negligible, it is assumed to be divided equally between the two ends. With

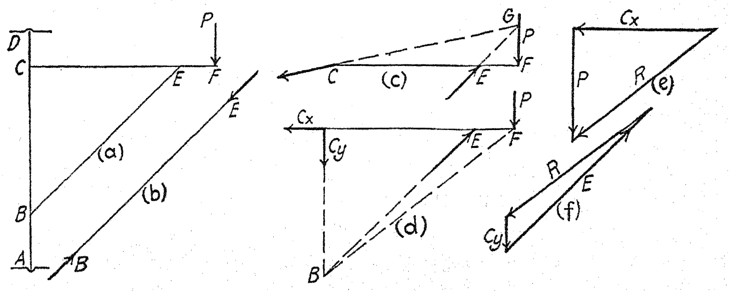


FIG. 85.

these assumptions, such members are called "two-force" members, since forces are applied to each one at only two points, the ends of the member. Since the resultant of the forces at one end must balance the resultant of those at the other, it is necessary that the line of action of these resultants shall be axial along the member. Since the stress is axial, a section may be made through such a member in taking a free body.

In the crane shown in Fig. 85(a), the brace BE is a two-force member if its own weight is negligible compared to the pressures at B and E. Its free-body diagram is shown in Fig. 85(b), although such a free-body diagram is never used in the solution of problems.

It is true that some bridge trusses are riveted at the joints and that some roof trusses are riveted at the joints and have some

of the members extending past several joints, but all are analyzed as consisting of two-force members unless the ability to take bending is necessary for stability. The criterion for stability without bending is that the members of the structure must form no figures but triangles, and external loads must be applied only at the vertices of the triangles.

If the members of a structure form figures other than triangles, or if loads are applied at points other than the vertices, some of the members have forces acting at three or more points and are called "multiple-force" members. The stress in such a member may be partly axial but is also bending and shear, so the entire member must be used as the free body. Solution is made for the pressures of other parts on the free body and not for the stress in the member.

If a body is in equilibrium under the action of three forces, they must intersect in a common point or be parallel, and the resultant of any two must be equal and opposite to the third. The boom CEF, Fig. 85(a), is a multiple-force member, shown as a free body in Fig. 85(c). The force at F is the external load P. That at E is equal and opposite to the force at E on the brace EB. Since these two are known in direction, their intersection at G determines another point on the line of action of the force at C. Likewise, if the entire crane is considered as a free body, the direction of the reaction at A is determined by the intersection of the lines of action of force P and the horizontal reaction at D.

If a body is in equilibrium under the action of *four* forces, the resultant of any two must necessarily be equal and opposite to the resultant of the other two. If one of the four forces is wholly known, and the directions of the others are known, the three unknown forces may be determined.

If the X and Y components of the reaction at C are used instead of the reaction itself, the boom just illustrated becomes an example of a system of four forces in equilibrium. The usual method of procedure would be to pair off forces P and E together, and C_x and C_y together. In this case the solution is identical with the solution of Fig. 85(c), with the addition of breaking up force C into its two components.

In order to vary the solution, C_x will be paired with P, and C_y with E. Line BF, Fig. 85(d), is then the common line of action of the resultants of the two pairs of forces. Figure 85(e) is the

solution of the force triangle at F, giving C_x and R the resultant of C_x and P. This resultant R must necessarily be in equilibrium with forces E and C_y, so these must form a closed triangle as shown in Fig. 85(f). If five or more forces are acting upon a body in equilibrium, two or more can usually be combined so as to reduce the system to a four-force system, after which it can be solved as above.

34. Redundant Force Systems.—If a structure has more members than are necessary for stability, such extra members are said to be *redundant.* If a section is made through such a redundant member in taking a free body, there will be more unknown quantities than there are independent conditions of equilibrium, and its force system is said to be *statically indeterminate.*

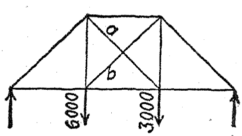


FIG. 86.

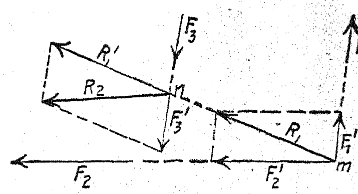


FIG. 87.

Structures may also be statically indeterminate if they have more external supports than are necessary for equilibrium. A table with four, five, six, or more legs of equal length resting on a level floor is an example. Since three legs are sufficient for the equilibrium of the table, any supports in excess of this number are redundant.

Sometimes a reasonable assumption can be made to obtain a solution. If a table has six legs equally spaced around the circumference of a circle, and if the loading is central, it is reasonable to assume that each leg carries one-sixth of the load. In the truss shown in Fig. 86, either a or b is a redundant member if both members can take either tension or compression. If the assumption is made that members a and b can take only tensile stress, b is the member acting, and a is not stressed when the truss is loaded as shown.

35. Resultant of Coplanar, Nonconcurrent Forces, Graphically. Let F_1, F_2, F_3, etc. (Fig. 87), be the forces to be combined. F_1 and F_2 are transferred along their lines of action until they intersect at m, where they are combined into their resultant R_1.

R_1 is transferred along its line of action until it intersects the line of action of F_3, at n. F_3 is also transferred along its line of action until it is in the proper position F_3', where it is combined with R_1 into the resultant of all three forces R_2. If there are other forces, the same procedure may be followed until the final resultant is found.

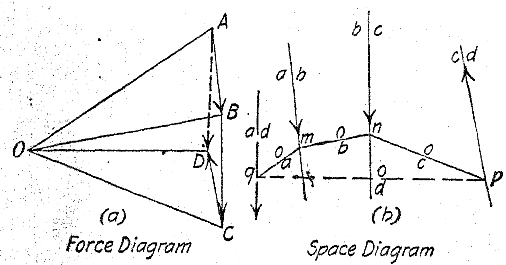


FIG. 88.

If the forces are so nearly parallel that no intersection can be obtained on the diagram, the same method may be employed as was used for parallel forces in Art. 27. In Fig. 88, forces AB, BC, and CD, acting along lines of action ab, bc, and cd, respectively, are combined into their resultant AD, acting along the line ad in the space diagram.

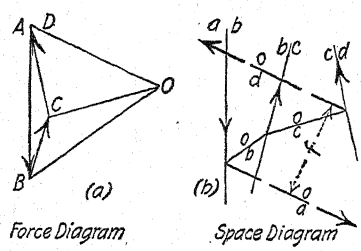


FIG. 89.

If the force polygon closes but the funicular polygon does not close, the resultant is a couple. In Fig. 89, force vectors AB, BC, and CD form a closed polygon, with point D coinciding with point A. These forces act along lines of action ab, bc, and cd, respectively, in the space diagram. From any point O, rays OA, OB, OC, and OD are drawn. In the space diagram the corresponding strings oa, ob, oc, and od are drawn. Strings oa and od are parallel but not collinear, so the system is reduced to the two equal parallel forces AO and OD acting f distance apart.

EXAMPLE 1

Figure 90(a) shows a boom upon which six forces are acting, four known and two unknown. Combine the four known forces into their resultant.

Solution.—In Fig. 90(*b*), the horizontal 6000-lb. force is shown transmitted along its line of action until it intersects the 4000-lb. force. Here the two are combined into their resultant R_1 = 7210 lb. The resultant of the other two 6000-lb. forces is necessarily 12,000 lb., acting halfway

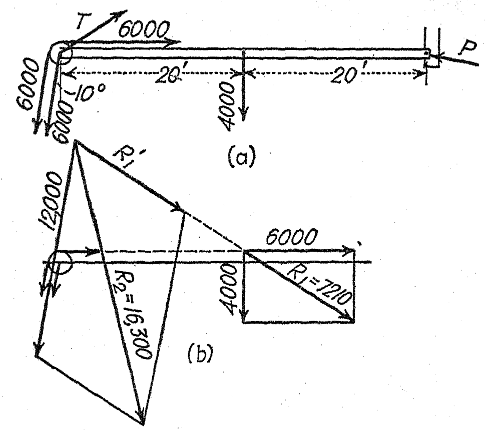


FIG. 90.

between the two. R_1 is transmitted backward along its line of action to position R_1' where it intersects the 12,000-lb. force. The two are then combined into their resultant R_2 = 16,300 lb., which is therefore the resultant of the original four known forces.

EXAMPLE 2

Combine into their resultant the wind- and dead-load forces acting upon the truss shown in Fig. 91(*a*).

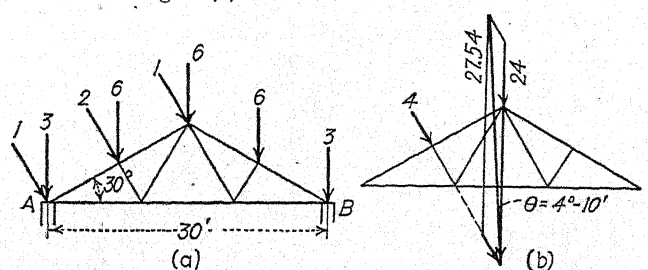


FIG. 91.

Solution.—It is evident from symmetry that the resultant of the five vertical forces is 24 kips acting at the middle of the truss. It is also evident that the resultant of the three wind loads is 4 kips, acting along the line of action of the 2-kip wind load. These are shown in position in Fig. 91(*b*). The completed parallelogram determines their resultant which scales 27.54 kips. Angle θ between the resultant and the vertical scales 4°10′.

Problems

1. Solve for the amount, direction, and position of the resultant of the loads on the bent shown in Fig. 92.

Ans. 22.56 kips, 77°10′ with the horizontal; R intersects BD 25.9 ft. from B.

2. Solve for the amount, direction, and position of the resultant of the seven loads on the dredge mast shown in Fig. 93. The pulley at A is 2 ft. in diameter, and those at B and C are each 1 ft. in diameter.

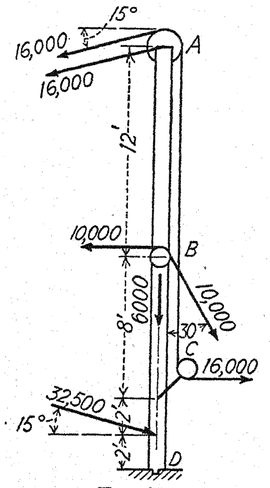


FIG. 93.

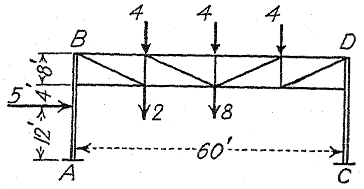


FIG. 92.

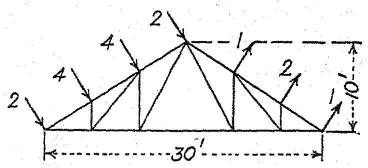


FIG. 94.

Ans. 33,390 lb. downward to the right, 69°55′ with the horizontal; R intersects the ground line at a point 21.5 ft. on the left of D.

3. Solve for the amount, direction, and position of the resultant of the wind loads on the truss shown in Fig. 94.

Ans. 11.1 kips, downward to the right, 36°50′ with the horizontal; R intersects the lower chord at a point 4.86 ft. from the left end.

36. Resultant of Coplanar, Nonconcurrent Forces, Algebraically.—For any system of coplanar, nonconcurrent forces, X and Y axes may be chosen. At any convenient point on its line of action, each force may be resolved into its X and Y components. The amounts of these components are independent of the position of these points of resolution. Since the translational effect of the resultant must be the same as that of its several components, the amount of the resultant is given by the expression

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

The direction angle θ of the resultant with the X axis is given by the expression

$$\theta = \tan^{-1} \frac{\Sigma F_y}{\Sigma F_x}$$

The position of the resultant with respect to any given point of reference will be determined by the principle of moments in Art. 37.

It is sometimes convenient to reduce a given system of coplanar, nonconcurrent forces to a single force through a given point and a couple. This may be done by extending the principle of Art. 32. Each force separately may be resolved into a force through the given point and a couple. The resulting system of concurrent forces at the given point may then be combined into their resultant R. Similarly, the several resulting couples may be combined by algebraic addition into a single resultant couple.

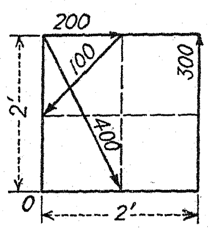


FIG. 95.

Problems

1. Determine the amount and direction of the resultant of the four forces shown in Fig. 95.

Ans. 334 lb.; $\theta = 337°20'$.

2. Reduce the 100-, 400-, and 300-lb. forces shown in Fig. 95 into a force through O and a couple.

Ans. 168 lb.; $\theta = 310°10'$; 313 lb.-ft.

37. Principle of Moments for Coplanar, Nonconcurrent Forces. As in the case of three or more parallel forces (Art. 28), the theorem of moments is readily extended to this case:

For any system of forces in a plane, the algebraic sum of the moments of the several forces with respect to any point is equal to the moment of the resultant with respect to the same point.

By means of this principle, the position of the resultant R may be determined by writing the equation of moments with respect to any point. If the moment arm of the resultant is denoted by a, and the moment arms of the several forces by a_1, a_2, etc.,

$$Ra = F_1a_1 + F_2a_2 + \cdots$$

EXAMPLE

Determine the normal distance of the resultant of the four forces shown in Fig. 95 from point O. Determine also the distance from O at which the resultant intersects the X axis.

Solution.—From Prob. 1, Art. 36, $R = 334$ lb. Assuming the resultant to act above point O, it will have negative moment, since it acts downward and to the right.

$$-334a = (300 \times 2) + (70.7 \times 1) - (200 \times 2) - (358 \times 1)$$
$$a = 0.262 \text{ ft.}$$

Let x be the distance from O at which the resultant intersects the X axis. Then distance $x = \dfrac{a}{\sin 22°40'} = 0.68$ ft.

The distance x can also be computed by using the equation $\Sigma M_0 = \Sigma F_y \times x$, since the X component acts along the X axis and therefore has no moment.

$$-128.7x = (300 \times 2) + (70.7 \times 1) - (200 \times 2) - (358 \times 1)$$
$$x = 0.68 \text{ ft., as before.}$$

Problems

1. In Fig. 95, solve for the amount, direction, and position of the resultant of the 100-, 200-, and 300-lb. forces.

Ans. $R = 263$ lb.; $\theta = 60°30'$; $a = 1.03$ ft.; $x = 1.18$ ft.

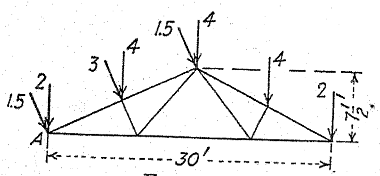


FIG. 96.

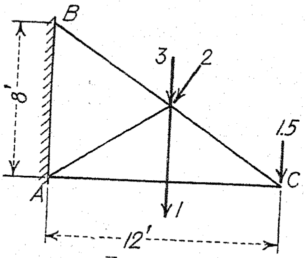


FIG. 97.

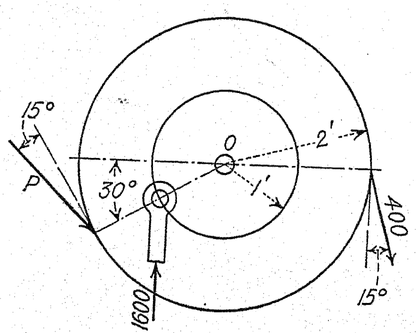


FIG. 98.

2. Solve for the amount, direction, and position of the resultant of the wind and dead loads on the truss shown in Fig. 96.

Ans. $R = 21.53$ kips; $\theta = 277°10'$; $a = 13.5$ ft.; $x = 13.6$ ft.

3. Solve for the amount, direction, and position of the resultant of the four forces shown in Fig. 97.

Ans. $R = 7.245$ kips; $\theta = 261°10'$; $a = 6.57$ ft.; $x = 6.65$ ft.

4. The gear wheel shown in Fig. 98 rotates about axis O. Solve for the amount of the force P, which has a moment with respect to O equal and opposite to the sum of the moments of the other two forces. Determine the amount, direction, and position of the resultant of the three forces.

Ans. $P = 1118$ lb.; $R = 989$ lb. acting through O; $\theta = 25°20'$.

38. Equilibrium of Coplanar, Nonconcurrent Forces: Graphic Solution.

—If for a system of coplanar, nonconcurrent forces the force polygon closes, the resultant R is zero. If also the funicular polygon closes, the moment M is zero, and the system is in equilibrium.

Conversely: If a system of coplanar, nonconcurrent forces is in equilibrium, the force polygon must close, and the funicular polygon must close.

The closing of the force polygon furnishes two independent conditions of equilibrium (corresponding to $\Sigma F_x = 0$ and $\Sigma F_y = 0$), and the closing of the funicular polygon furnishes a third independent condition (corresponding to $\Sigma M = 0$), so unknown elements not exceeding three can be determined. These unknown elements may be the amounts of three unknown forces, the directions being known, or they may be the amounts of two forces and the direction of one of them, the direction of the other being known.

In case the known forces can be combined conveniently into their resultant, the system is reduced to a coplanar, concurrent system; for through the point where this resultant meets one of the unknown forces, the remaining unknown force (or the resultant of the remaining two) must act.

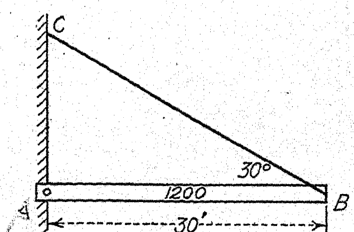


FIG. 99.

EXAMPLE 1

The horizontal boom AB shown in Fig. 99 is supported by a pin at A and a tie rod BC. Solve for the tension in the tie rod BC and the amount and direction of the pin reaction at A.

Solution.—The free-body diagram is shown in Fig. 100(a). The weight of the boom is a vertical downward force of 1200 lb. at the center of gravity. The tension in the tie rod BC acts upward to the left from point B. The pin reaction at A must pass through point A and also through the inter-

section of the other two forces at D. By symmetry, angle $\theta = 30°$. The force triangle is shown in Fig. 100(b), from which stress BC and reaction A each scales 1200 lb.

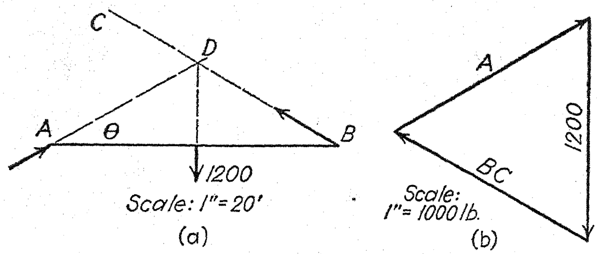


FIG. 100.

EXAMPLE 2

The cantilever truss represented in Fig. 101 is pin-connected and carries five loads as shown. Solve for the reactions at A and B.

Solution.—The free-body diagram is shown in Fig. 102(a). MN is selected as a base line, from which MP is laid off 2 kips to scale and NQ is laid off 3 kips to scale. Line PQ intersects MN at O, thus locating the line of action of their resultant, 5 kips in amount. By symmetry, the resultant of the three wind loads is 4 kips, acting at point D. This force and the 5-kip force intersect at R. At this point, vectors RS = 4 kips and RT = 5 kips are laid off to scale, and the resultant RU = 8.81 kips is obtained.

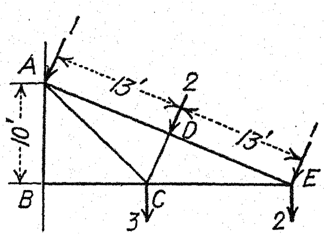


FIG. 101.

The five known forces have now been reduced to one. The reaction at B must be acting along the axis of the member BC. These two forces inter-

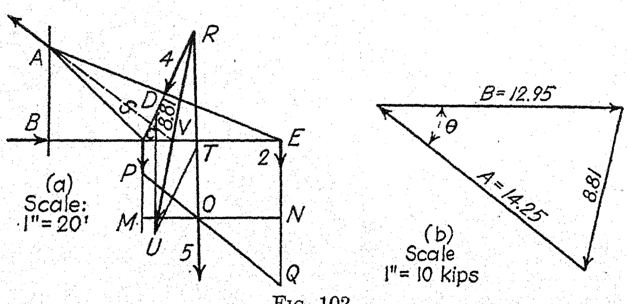


FIG. 102.

sect at point V, so the reaction at A, in order to balance these two, must also pass through point V. The solution of the force triangle for these forces is shown in Fig. 102(b), from which the reactions are scaled as indicated. The angle θ between reaction A and the horizontal scales 38°.

EXAMPLE 3

Solve for the reactions GH and HA of the cantilever truss shown in Fig. 103.

Solution.—In Fig. 104(*a*), the space diagram is laid off to the scale 1 in. = 20 ft. In Fig. 104(*b*), the force diagram is laid off as far as known to the scale 1 in. = 8 kips. It is known that GH and HA must close the force polygon, but the location of point H along the line GH is unknown. The pole O is selected, and rays OA, OB, etc., are drawn. In the space diagram, the funicular polygon is drawn, beginning at the upper hinge p, since it is the only point known on the reaction ha. The string oa is zero length, since it is drawn between ha and ab. The strings ob, oc, od, oe, of, and og, parallel, respectively, to OB, OC, etc., are drawn. The string og intersects gh at q. Since the system is in equilibrium, it is necessary for the funicular polygon to close; hence string oh must be drawn from point q to

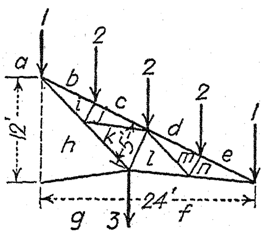


FIG. 103.

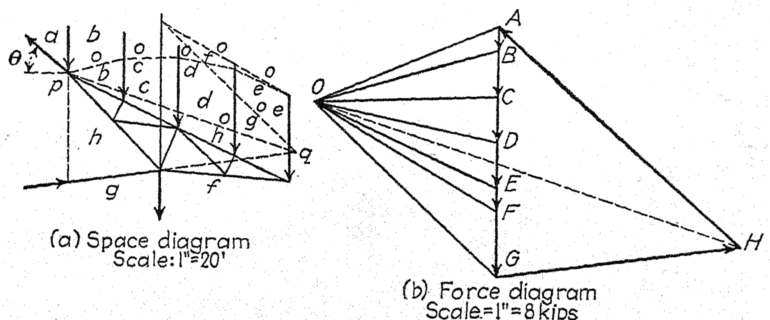


FIG. 104.

point p. The line OH in the force diagram, parallel to oh in the space diagram, locates point H and so determines the amount of reaction GH and the amount and direction of reaction HA.

GH scales 10.57 kips; HA scales 14 kips; and angle θ scales 42°.

Problems

1. In Fig. 105, a block 4 ft. long and 1 ft. square, weighing 640 lb., is held in a horizontal position by force P and the reaction of the inclined plane. Solve for force P and for the normal and tangential components of the reaction at A.

Ans. P = 310 lb.; A_N = 335 lb.; A_T = 101 lb.

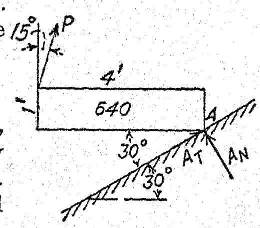


FIG. 105.

2. In the crane shown in Fig. 106, length AB = 40 ft., AC = 60 ft., and CD = 20 ft. The weight of the boom is 3200 lb., with its center of

gravity 30 ft. from A. Solve for the stress in BC and for the vertical and horizontal components of the reaction at A.

Ans. $BC = 13{,}200$ lb.; $A_V = 20{,}930$ lb.; $A_H = 6550$ lb.

3. The cantilever truss shown in Fig. 107 is pin-connected at all joints. Solve for the reaction at B and for the vertical and horizontal components of the reaction at A.

Ans. $B = 16.7$ kips; $A_V = 19.46$ kips; $A_H = 14.7$ kips.

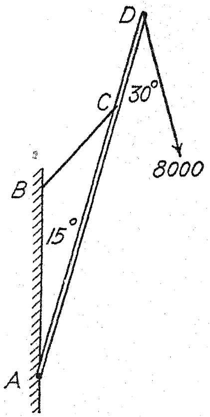


FIG. 106.

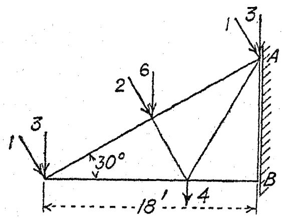


FIG. 107.

39. Equilibrium of Coplanar, Nonconcurrent Forces: Algebraic Solution.—If the resultant R of a system of coplanar, nonconcurrent forces is zero, and the moment M with respect to any axis normal to the plane of the forces is also zero, the effect of the system is zero, and the system is in equilibrium.

Conversely: If a coplanar, nonconcurrent system of forces is in equilibrium, the resultant R is equal to zero, and the moment with respect to any point in the plane is equal to zero.

Since the resultant is equal to zero, the summation of forces along any axis is equal to zero.

$$\Sigma F_x = 0, \ \Sigma F_y = 0, \ \Sigma M = 0$$

These three independent equations permit the determination of three unknown elements. These three unknown elements very often consist of the amounts of three of the forces, their directions being known. If two of the unknown forces are parallel, the equation of the summation of forces normal to these will give the value of the third unknown force. In any case, the equation of moments with respect to the intersection of two of the unknown forces will give the third unknown force.

In some cases, the three unknown elements consist of the amount and direction of one of the forces and the amount of another, its direction being known. The moment equation with respect to the known point of the force which is unknown both in amount and direction will determine the other unknown

force. The force that is unknown in both amount and direction may be replaced by two rectangular components, thus reducing this to the same case as the preceding.

EXAMPLE 1

Solve for the reactions at C and E on the cantilever truss shown in Fig. 108(a). The truss is pin-connected at all joints.

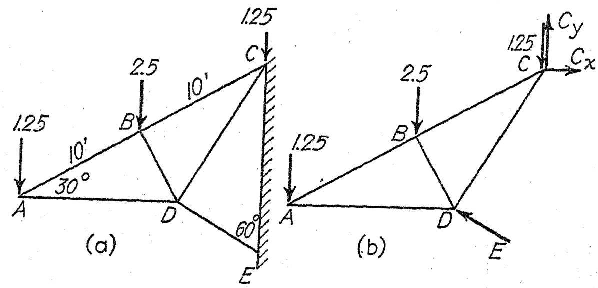


FIG. 108.

Solution.—Since member DE is a two-force member, the stress in it is axial, and the reaction at E is the same as the stress in DE, both in amount and in direction. The reaction at C is replaced by its horizontal and vertical components C_x and C_y. The equation $\Sigma M = 0$ with respect to point C gives

$$11.55E - (1.25 \times 17.32) - (2.5 \times 8.66) = 0$$
$$E = 3.75 \text{ kips}$$

The equation $\Sigma F_x = 0$ gives

$$C_x - (3.75 \times 0.866) = 0$$
$$C_x = 3.25 \text{ kips}$$

The equation $\Sigma F_y = 0$ gives

$$C_y + (3.75 \times 0.5) - 5 = 0$$
$$C_y = 3.125 \text{ kips}$$
$$C = \sqrt{3.25^2 + 3.125^2} = 4.51 \text{ kips}$$

Let θ be the angle that reaction C makes with the horizontal.

$$\tan \theta = \frac{3.125}{3.25} = 0.96$$
$$\theta = 43°50'$$

EXAMPLE 2

The A-frame shown in Fig. 109(a) supports a load of 8000 lb. at the middle of member BD. Solve for the pin reactions at B, C, and D caused by this load, if the supporting floor is considered smooth.

Solution.—The entire frame is the first free body. Since the floor is smooth, the reactions at A and E are necessarily vertical. The frame and loading are symmetrical, so each reaction is 4000 lb. If either the frame or the loading were not symmetrical, the moment equation with respect to a point on one reaction would determine the other.

All the members are multiple-force members, so each member must be taken in its entirety as a free body. The cross bar BD is taken next, and its free-body diagram is shown in Fig. 109(*d*). The known force is 8000 lb. acting downward at the middle. Since a multiple-force member joins it

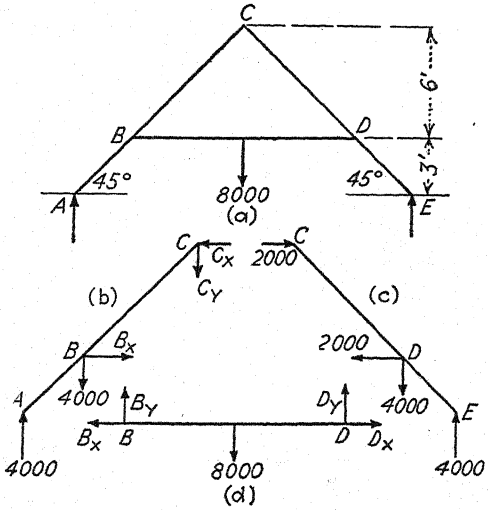


FIG. 109.

at B, and another at D, the vertical and horizontal components of the reactions at these points must be used. Either by the equation of moments or by symmetry,

$$B_y = D_y = 4000 \text{ lb.}$$

The equation $\Sigma F_x = 0$ gives $B_x = D_x$, but since they are collinear, they cannot be evaluated from this free body.

Member AC is taken as the next free body, and its free-body diagram is shown in Fig. 109(*b*). On this free body, forces A and B_y are known, so their values are placed on the vectors. Forces B_x, C_x, and C_y are the unknown forces. The equation $\Sigma F_y = 0$ gives

$$C_y + 4000 - 4000 = 0$$
$$C_y = 0$$

If the frame were not symmetrical and symmetrically loaded, the value of C_y, in general, would not be zero.

The equation $\Sigma M_B = 0$ gives

$$(C_x \times 6) - (4000 \times 3) = 0$$
$$C_x = 2000 \text{ lb.}$$

The equation $\Sigma F_x = 0$ gives

$$B_x = C_x = 2000 \text{ lb.}$$

Also, on member BD,

$$D_x = B_x = 2000 \text{ lb.}$$

Since C_y is zero, the pin reaction at C is 2000 lb. horizontal. The pin reactions at B and D are given by the equation

$$B = D = \sqrt{2000^2 + 4000^2} = 4472 \text{ lb.}$$

The angle θ of each reaction with the horizontal is given by the equation

$$\theta = \tan^{-1}\frac{4000}{2000} = 63°30'$$

All the forces acting on member CE are now known, as shown in Fig. 109(c), and the three equations of equilibrium may be applied to it as a check on the solution.

Problems

1. The boom AB shown in Fig. 110 weighs 400 lb. with its center of gravity at the middle, is pin-connected to the wall at B, and supported by cable AC. It also supports a 2000-lb. pull at A as shown. Solve for

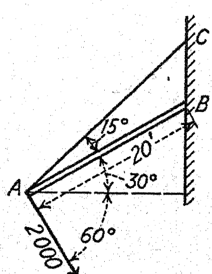


FIG. 110.

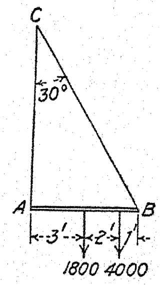


FIG. 111.

the stress in AC and for the vertical and horizontal components of the pin reaction at B.

Ans. $AC = 8400$ lb.; $B_x = 6940$ lb.; $B_y = 3810$ lb. downward.

2. The supporting crossbar AB of the platform shown in Fig. 111 is hinged to the wall at A, supported by the tie rod BC, and carries loads as shown. Solve for the stress in BC and for the amount and direction of the hinge reaction at A. *Ans.* $BC = 4890$ lb.; $A = 2905$ lb.; $\theta = 32°40'$.

3. The cantilever truss shown in Fig. 112 is assumed to be pin-connected at all joints. Solve for the reaction at B and for the horizontal and vertical components of the reaction at A.

Ans. $B = 25.3$ kips; $A_x = 23.23$ kips; $A_y = 17.73$ kips.

4. Solve for all the reactions on the A-frame shown in Fig. 109 if the 8000-lb. load is moved 4 ft. to the left.

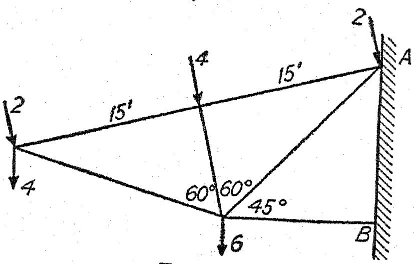


FIG. 112.

Ans. $A = 5780$ lb.; $E = 2220$ lb.; $B = 6960$ lb., 73°20′ with X; $D =$ 2400 lb., 33°40′ with X; $C = 2190$ lb., 24° with X.

5. Solve for all the reactions on the A-frame shown in Fig. 109 if the 8000-lb. load is removed, the diagonal members weigh 100 lb./ft. and the horizontal member weighs 150 lb./ft.

Ans. $A = 2170$ lb.; $B = 1665$ lb., 32°40′ with X; $C = 1400$ lb. horizontal.

6. The frame shown in Fig. 113 is pinned at A and E, held by the tie bar CD, and supported against a smooth ceiling at B. Solve for the reactions A, B, and E, and for the stress in CD caused by the 1200-lb. load.

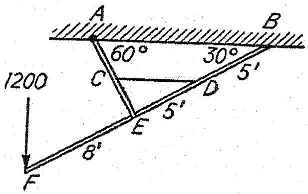


FIG. 113.

Ans. $A = 1620$ lb.; $B = 420$ lb.; $E = 2475$ lb., 40°55′ with X; $CD =$ 1870 lb. C.

40. Stresses in Trusses: Graphic Solution.—As explained in Art. 33, roof and bridge trusses are commonly assumed to be composed of two-force members in which the stresses are axial. In simple trusses, it is necessary to solve for at least one reaction before the solution for the internal stresses can be begun. If the loads and reactions are parallel, the method of Art. 29 is to be used. If they are not parallel, the method of Art. 38 is to be used. In cantilever trusses, solution for the internal stresses may be made without first solving for the reactions. In either case, each joint in turn is taken as a free body, and its force polygon is drawn to scale. From the fact that the force polygon

must close, the unknown forces, not to exceed two, can be determined.

If at any joint there are three unknown forces, solution cannot be made until the value of one of the unknown forces is obtained by using several of the joints together as the free body. If the system of forces on this free body is nonconcurrent, three unknown forces may be determined by the method of Art. 38. If it is not convenient to combine all the known forces into a single resultant, the solution may be made in two or more parts. The vector sum of the partial stresses obtained in this manner will be the true stress in the member. Since the unknown forces at one of the joints have now been reduced to two, solution may be continued joint by joint.

In drawing the polygon for any joint, one or more of the forces have been determined in previous solutions, and the vectors are in place. If these are used in the new solution, the force polygons are linked together to form the *stress diagram*. If Bow's notation is used, the diagram is self-checking. The direction in which any force acts on the joint is given by the direction of the vector between the letters by which it is designated in the force diagram. For example, if a force acts along line *cd* in the space diagram, reading clockwise around the joint, the direction from *C* to *D* in the force diagram shows the direction of the force in *cd* on the joint.

EXAMPLE 1

Solve for the stresses in the members of the cantilever truss shown in Fig. 114(*a*). The truss is 30 ft. long and 10 ft. deep at the supporting wall.

Solution.—Since this is a cantilever truss, it is not necessary to solve for the reactions first. The first joint at the left end of the truss may be used as the free body. At the left end of Fig. 114(*b*) is shown the method of representing the free body. The arrowhead on member *be* is shown acting to the right away from the joint, the action of the member on the pin. The arrowhead on member *ea* is shown acting upward to the left toward the joint, the action of member *ea* on the pin.

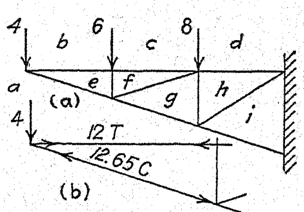


Fig. 114.

The force polygon for this joint is shown in Fig. 115(*a*). Vector *AB* is laid off to scale, 4 kips downward. It is necessary for the force polygon to close from point *B* back to point *A* with vectors parallel to lines *be* and *ea*.

(The forces are taken in order clockwise around the joint, and the sequence of letters by which the line of action of a force is known is always clockwise.)

The scaled value of BE is 12 kips. The stress is tension, since the direction from B to E is from left to right and therefore indicates a force from left to right in member *be* on the joint *abe*. The scaled value of EA is 12.65 kips, and the direction from E to A indicates compression in member *ea*. As soon as the direction of the stress in a member is determined, arrowheads should be placed at each end of it showing its action on the

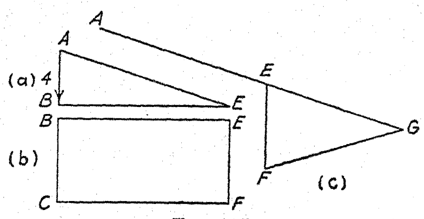


FIG. 115.

joints it connects. Figure 114(*b*) shows to larger scale the left end of the space diagram after the stresses in *be* and *ea* have been determined.

The next free body is the second joint on the upper chord, joint *ebcf*. Forces EB and BC are known and are laid down in order in Fig. 115(*b*). Vector CF parallel to *cf* and vector FE parallel to *fe* must close the polygon from C to E. Stress CF is 12 kips tension, and stress FE is 6 kips compression. It is plain that line BE in the second polygon could have been made coincident with BE in the first polygon, linking the two together.

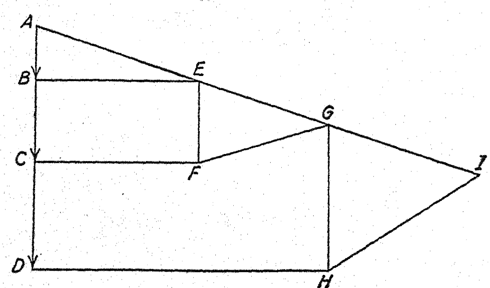


FIG. 116

On the next joint, *aefg*, forces AE and EF are already known, so vector FG parallel to *fg* and vector GA parallel to *ga* must close from F to A. The polygon is shown in Fig. 115(*c*). Vector FG scales 9.49 kips tension, and vector GA scales 22.1 kips compression. As before, this polygon could have been linked with the others. Figure 116 shows the complete solution for all five joints of the truss, with the polygons all linked together to form one stress diagram. Vector DH scales 21 kips tension; vector HG scales 11 kips compression; vector HI scales 13.2 kips tension; and vector IA scales 33.7 kips compression.

EXAMPLE 2

In the truss shown in Fig. 117, solution can be made with no difficulty for the first three joints at either end. The next joints have each three unknown

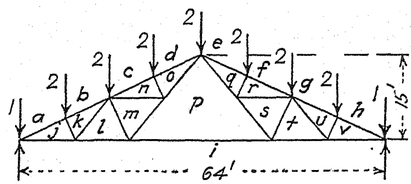


FIG. 117.

forces. With the left half of the truss as a free body, solve for the stress in member *pi* so that joint *pilm* can be solved after the stresses up to this joint have been obtained.

Solution.—By symmetry, each reaction is 8 kips. The load on each half of the truss is also 8 kips, so the resultant of the system of known forces on the left half of the truss is a couple. The system to balance this must also be a couple; so since the force in *pi* is horizontal, the hinge reaction at the upper point of the truss must also be horizontal. The free-body diagram for the left half of the truss is shown in Fig. 118, with the vertical loads replaced by their resultant. The system of four forces must be in equilibrium. The force in *ep* and the upward reaction meet at A, and their resultant must balance the resultant of the force in *pi* and the downward load which meet at B, so AB is the required direction of the two resultants. Vector BD, 8 kips to scale, is laid off to represent the downward vertical load. Through point D, a line is drawn parallel to *pi* which intersects AB produced at C. Vector DC gives the required force in *pi* and scales 8.53 kips. The force in *ep* is, of course, equal in amount and oppositely directed.

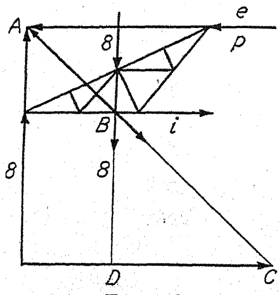


FIG. 118.

Problems

1. Using the value for the stress in *pi* obtained in Example 2, solve for all the internal stresses in the left half of the truss shown in Fig. 117.

Ans. AJ = 16.5 kips C; JI = 14.93 kips T; BK = 15.65 kips C; KJ = 1.81 kips C; KL = 2.15 kips T; LI = 12.83 kips T; LM = 3.62 kips C; MP = 4.3 kips T; CN = 14.8 kips C; NM = 2.15 kips T; DO = 13.95 kips C; ON = 1.81 kips C; OP = 6.4 kips T.

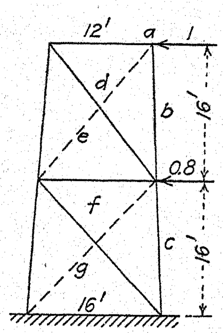


FIG. 119.

2. In the tower shown in Fig. 119, the diagonal members represented by dotted lines are assumed to carry no stress when loads are applied on the right-

hand side. Assuming the tower to be pin-connected at all joint librium hori- the internal stresses.

Ans. $AD = 1$ kip C; $AE = 1.14$ kips C; $DE = 1.47$ kips ; $EF = 1.66$ kips C; $AG = 2.80$ kips C; $FG = 2.27$ kips T; $FC = 1.$

3. Solve for the reactions R_1 and R_2, and for all the internal str the truss shown in Fig. 120.

Ans. $R_1 = 21$ kips; $R_2 = 10$ kips; $AG = 18.97$ kips T; $GB = 18$ kips C, $GH = 1$ kip T; $AI = 20.55$ kips T; $IH = 1.58$ kips C; $HC = 18$ kips C; $IJ = 2.50$ kips T; $JD = 19.5$ kips C; $AK = 23.19$ kips T; $KJ = 3.01$

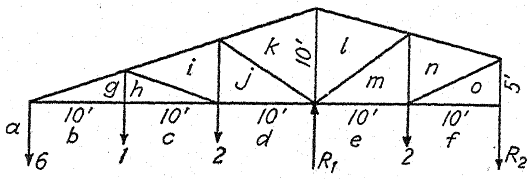


FIG. 120.

kips C; $AL = 22.70$ kips T; $LK = 12.84$ kips C; $LM = 10.88$ kips C; $ME = 13.32$ kips C; $AN = 13.74$ kips T; $NM = 8.68$ kips T; $NO = 14.91$ kips C; $OF = 0$.

41. Stresses in Trusses: Algebraic Solution.—The stresses in cantilever trusses may be determined without solving for the reactions; but in the case of simple trusses, it is necessary to solve for the reactions first, either by the method of this chapter or by that of Chap. III. In making solution for the internal stresses in the members, each joint in turn may be taken as the free body except in special cases. The forces acting at the joint constitute a coplanar, concurrent system in equilibrium, and solution is made by one of the methods of Chap. II.

After the free body has been isolated, the direction of each unknown stress can often be determined by observation. If in any case it cannot be thus determined, a direction should be assumed, and the arrowhead placed on the vector of the force. If the solution of the equations of equilibrium gives a positive value to the stress, the correct direction was assumed, whereas if it gives a negative value the wrong direction was assumed, and the arrowhead must be changed.

In case there are three unknown forces at any joint, solution cannot be made, since there are only two independent conditions of equilibrium. If no joint remains with less than three unknown forces, a free body including several joints may be selected in

that the system is a nonconcurrent system with no three unknown forces. With this free body, one of

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own forces can be determined, after which the solution may be continued in the regular way, joint by joint.

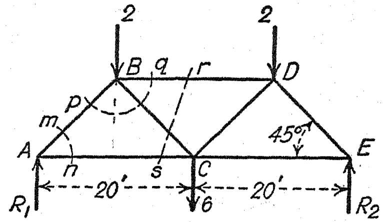


FIG. 121.

EXAMPLE 1

Determine all the internal stresses in the members of the pin-connected bridge truss shown in Fig. 121.

Solution.—The truss is symmetrical, and the loading is symmetrical, so each reaction is one-half the total load of 10 kips.

$$R_1 = R_2 = 5 \text{ kips}$$

All the members are two-force members, so sections may be made through them as desired in taking a free body. Let a section be made through the truss at *mn*, and let the part at *A* be taken as the free body, as shown in Fig. 122(*a*). The force F_1 is the internal stress in *AB*, acting now as an external force on the free body, and must be compression in order to balance R_1. Similarly, the force F_2 is the internal stress in *AC* and must be tension in

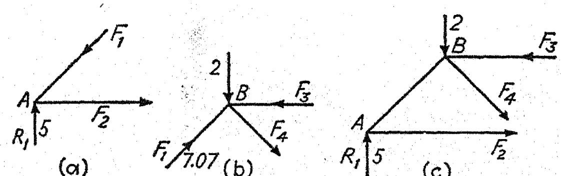


FIG. 122.

order to balance F_1. Since this free body is in equilibrium, $\Sigma F_y = 0$, and $\Sigma F_x = 0$. Equation $\Sigma F_y = 0$ gives

$$5 - F_1 \sin 45° = 0$$
$$F_1 = 7.07 \text{ kips compression}$$

Equation $\Sigma F_x = 0$ gives

$$F_2 - 7.07 \cos 45° = 0$$
$$F_2 = 5 \text{ kips tension}$$

The next free body taken is the joint at *B*, enclosed by section *pq*. The free-body diagram is shown in Fig. 122(*b*). There are two known forces acting on the free body, the load (2 kips) and the stress in *AB*. The action of this force on B must be equal in amount and opposite in direction to the force that *AB* exerts on *A* so is 7.07 kips, acting as shown. Inspection of the known vertical forces acting at *B* shows that there is a larger force upward than downward. Therefore, for equilibrium, F_4 must have a

component downward, so is tension, as shown. For equilibrium horizontally, F_3 must be compression. Equation $\Sigma F_y = 0$ gives

$$7.07 \sin 45° - 2 - F_4 \sin 45° = 0$$
$$F_4 = 4.24 \text{ kips tension}$$

Equation $\Sigma F_x = 0$ gives

$$7.07 \cos 45° + 4.24 \cos 45° - F_3 = 0$$
$$F_3 = 8 \text{ kips compression.}$$

It is not necessary that the true direction of the unknown stresses be determined before solution, as above. If in either of the force diagrams the direction of an unknown force had been assumed incorrectly, the value obtained would have been the same numerically but negative in sign.

Since the truss and loading are both symmetrical, the stresses in corresponding members on the two sides of the truss are equal, so the solution need not be carried further unless it is desired to complete it as a check.

If the stress in only one member had been required, as, for instance, that in BD, a shorter method would have been as follows: Let the section rs, Fig. 121, be passed through the truss, and let all the truss on the left of the section be taken as the free body. Figure 122(*c*) shows the free-body diagram. There are now three unknown forces, but they are not concurrent, so the problem can be solved. Equation $\Sigma M_C = 0$ gives

$$(F_3 \times 10) - (5 \times 20) + (2 \times 10) = 0$$
$$F_3 = 8 \text{ kips compression}$$

Stresses F_2 and F_4 can now be determined if desired. Equation $\Sigma F_y = 0$ gives

$$F_4 \sin 45° + 2 - 5 = 0$$
$$F_4 = 4.24 \text{ kips}$$

Equation $\Sigma M_B = 0$ gives

$$(F_2 \times 10) - (5 \times 10) = 0$$
$$F_2 = 5 \text{ kips}$$

EXAMPLE 2

Solve for the stresses in the members of the truss shown in Fig. 123. Point J is the middle point between I and D.

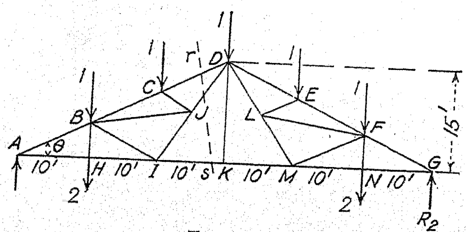


FIG. 123.

Solution.—The truss is symmetrical and is loaded symmetrically, so each reaction is one-half the total load of 9 kips.

$$R_1 = R_2 = 4.5 \text{ kips}$$

Length BH is 5 ft. and length AB is equal to $\sqrt{10^2 + 5^2} = 11.18$ ft.

$$\sin\theta = \frac{5}{11.18} = 0.447$$
$$\cos\theta = \frac{10}{11.18} = 0.894$$

The joint at A is the first free body. The equation $\Sigma F_y = 0$ gives

$$0.447\,AB - 4.5 = 0$$
$$AB = 10.06 \text{ kips } C$$

Equation $\Sigma F_x = 0$ gives

$$AH - 10.06 \times 0.894 = 0$$
$$AH = 9 \text{ kips } T$$

Joint H is the next free body. Equation $\Sigma F_x = 0$ gives

$$HI = 9 \text{ kips } T$$

Equation $\Sigma F_y = 0$ gives

$$HB = 2 \text{ kips } T$$

At both joints B and I there are three unknown forces, so neither joint can be solved. If section rs is passed through the truss and all the truss

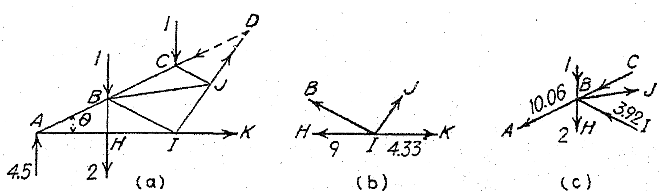


FIG. 124.

on the left of the section is taken as the free body, solution can be made for one of the stresses at the section. Fig. 124(a) shows the free-body diagram. For this free body, the equation $\Sigma M_D = 0$ gives

$$15IK + (1 \times 10) + (3 \times 20) - (4.5 \times 30) = 0$$
$$IK = 4.33 \text{ kips } T$$

The number of unknown forces at I is now reduced to two and solution can be made. The free-body diagram is shown in Fig. 124(b). The directions of stresses IB and IJ are unknown, so both are assumed to be tension.

$$\sin JIK = 0.833$$
$$\cos JIK = 0.555$$

Angle HIB is the same as angle θ. Equation $\Sigma F_x = 0$ gives

$$9 + 0.894IB - 0.555IJ - 4.33 = 0$$

Equation $\Sigma F_y = 0$ gives

$$0.447IB + 0.833IJ = 0$$

The solution of these two equations gives

$$IB = -3.92 \text{ kips}$$
$$IJ = +2.1 \text{ kips}$$

The positive sign obtained for IJ indicates that the assumption of tension was correct. The negative sign obtained for IB indicates that the direction assumed was incorrect, that the stress is really compression.

At joint B there are now only two unknown forces. Since force IB is known to be compression, it is shown acting in its true direction in the free-body diagram, Fig. 124(b). Upper chords of simple trusses with vertical loads are always in compression, so the direction of stress BC is known. The direction of the stress in BJ is determined by observation of joints C and J. At joint C, CJ is necessarily in compression to balance the component of the load normal to BC. Then at J, stress BJ is necessarily tension to balance the compression in CJ. The equation $\Sigma F_x = 0$ gives

$$(10.06 \times 0.894) + 0.988JB - (3.92 \times 0.894) - 0.894BC = 0$$

The equation $\Sigma F_y = 0$ gives

$$3 + 0.447BC - (10.06 \times 0.447) - (3.92 \times 0.447) - 0.1644JB = 0$$

The solution of these two equations gives

$$JB = 1.55 \text{ kips } T$$
$$BC = 7.85 \text{ kips } C$$

In a similar manner, solution of joints C and J gives the following values:

$$CD = 6.73 \text{ kips } C$$
$$CJ = 1.12 \text{ kips } C$$
$$JD = 3 \text{ kips } T$$

By inspection, stress $DK = 0$. By symmetry, the stress in any member in the right half of the truss is the same as that in the corresponding member in the left half.

Problems

1. Solve for the stresses in all the members of the cantilever truss shown in Fig. 125.

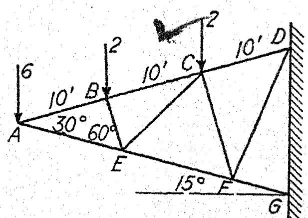


FIG. 125.

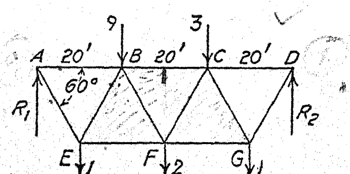


FIG. 126.

Ans. AB = 11.59 kips T; AE = 11.59 kips C; BE = 1.93 kips C; BC = 12.11 kips T; EC = 1.93 kips T; EF = 13.52 kips C; CF = 2.90 kips C; CD = 14.30 kips T; FD = 2.56 kips T; FG = 15.46 kips C.

2. Solve for the stresses in all the members of the simple truss shown in Fig. 126.

Ans. AB = 5.2 kips C; AE = 10.4 kips T; BE = 9.25 kips C; EF = 9.83 kips T; BF = 1.16 kips C; BC = 9.25 kips C; FC = 3.46 kips T; FG = 7.52 kips T; CG = 6.93 kips C; CD = 4.04 kips C; GD = 8.08 kips T.

3. Solve for the stresses in the members in the left half of the truss shown in Fig. 127. The loads are equally spaced across the truss.

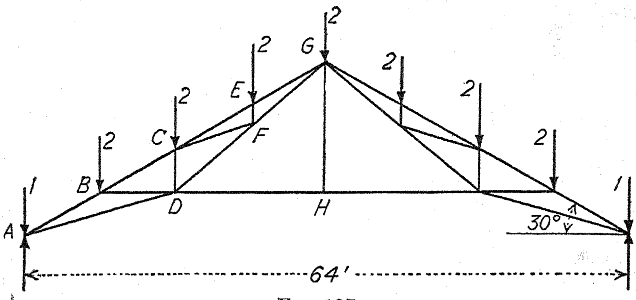


FIG. 127.

Ans. AB = 28.05 kips C; AD = 25.25 kips T; BC = 24.05 kips C; BD = 3.46 kips C; CD = 2.98 kips C; CE = 28.0 kips C; CF = 3.62 kips T; DF = 15.5 kips T; EF = 2 kips C; EG = 28.0 kips C; FG = 20.0 kips T; DH = 9.24 kips T; GH = 0.

42. Stresses in Bents: Algebraic Solution.—A *bent* of a mill building consists of two columns between which is framed the

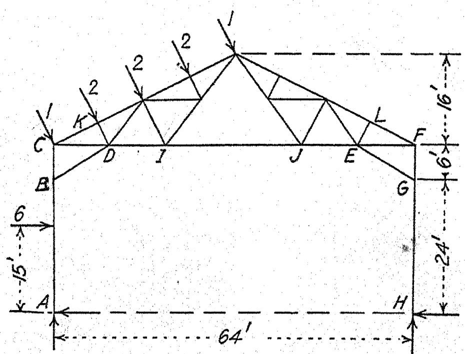


FIG. 128.

roof truss, as shown in Fig. 128. Members ABC and HGF are columns, connected to the roof truss by pins at the upper ends and by the knee braces BD and EG. The knee braces and the members of the roof truss are all two-force members, but the columns are multiple-force members, since they must take bend-

ing stresses for all except direct vertical loads. Columns are considered to be either (1) pin-connected or (2) fixed at the base. If fixed at the base, a lateral load will deflect the bent in the direction of the load, but CB and GF will remain vertical, and the tangents to the columns at A and H will remain vertical. Under these conditions, the points of counterflexure will be midway between A and B and between H and G. These points of counterflexure are equivalent to the bases of the pin-connected columns, so columns with fixed ends will be solved as though hinged at a point midway between the base and the point of attachment of the knee brace.

The distribution of the horizontal components of the reactions between the two columns is indeterminate, so some assumption must be made. The two common assumptions are (1) that each column takes one-half the horizontal component of the reaction and (2) that one takes all the horizontal component of the reaction (the other none). The latter assumption is the safer in design.

In solving for the reactions and stresses in a bent, the entire bent is the first free body, from which the reactions at the bases of the columns can be determined. Either one of the columns is the next free body, from which the stresses in the knee brace and the two members at the upper end of the column may be obtained. From this point on, the solution is the same as that for a truss resting on walls at the ends.

EXAMPLE

In the bent shown in Fig. 128, solve for the horizontal and vertical components of the reactions, for the stresses in the knee braces, and for the stresses in members CK, CD, and IJ. Assume the horizontal components of the reactions at A and H to be equal.

Solution.—The resultant of the diagonal wind loads on the roof is 8 kips, acting at the middle point. The horizontal component of this 8-kip force is $8/2.236 = 3.58$ kips, so the total horizontal load is $6 + 3.58 = 9.58$ kips.

$$A_x = H_x = \frac{9.58}{2} = 4.79 \text{ kips}$$

The vertical component of the load on the roof is $8 \times 2/2.236 = 7.16$ kips. The equation $\Sigma M_A = 0$, using the vertical and horizontal components of the load on the roof, gives

$$64H_y - (6 \times 15) - (3.58 \times 38) - (7.16 \times 16) = 0$$
$$H_y = 5.32 \text{ kips}$$

The equation $\Sigma F_y = 0$ gives

$$A_y + 5.32 - 7.16 = 0$$
$$A_y = 1.84 \text{ kips}$$

The next free body used is the column ABC. Its free-body diagram is shown in Fig. 129(a). There are three unknown forces on the free body; but since they are nonconcurrent, solution can be made. The equation $\Sigma M_C = 0$ gives

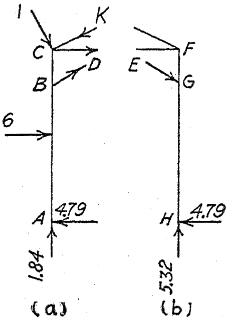


FIG. 129.

$$(BD \times 5.15) + (6 \times 15) - (4.79 \times 30) = 0$$
$$BD = 10.4 \text{ kips } T$$

The equation $\Sigma F_y = 0$ gives

$$1.84 + \left(\frac{6}{11.66}\right)10.4 - \frac{2}{2.236} - \frac{CK}{2.236} = 0$$

$$CK = 14.1 \text{ kips } C$$

The equation $\Sigma F_x = 0$ gives

$$CD + \left(10.4 \times \frac{10}{11.66}\right) - 4.79 + 6 + \frac{1}{2.236} - \left(14.1 \times \frac{2}{2.236}\right) = 0$$

$$CD = 2.0 \text{ kips } T$$

The column FGH is the next free body, shown in Fig. 129(b). The equation $\Sigma M_F = 0$ gives

$$(EG \times 5.15) - (4.79 \times 30) = 0$$
$$EG = 27.9 \text{ kips } C$$

The simplest free body to use in solving for the stress in member IJ is the right half of the bent. Assuming the stress to be tension, the equation of moments with respect to the upper point on the bent gives

$$(IJ \times 16) + (4.79 \times 46) - (5.32 \times 32) = 0$$
$$IJ = -3.13 \text{ kips}$$

Since the sign of the stress is negative, the direction assumed was wrong, and the stress in IJ is compression.

Problems

1. Solve for the stresses in members EF and LF, Fig. 128.

Ans. $EF = 1.1$ kips T; $LF = 20.2$ kips T.

2. Solve for the stresses in the knee braces BD and EG (Fig. 128), assuming (1) that all the horizontal reaction is carried at A and (2) that all of it is carried at H.

Ans. (1) $BD = 38.3$ kips T; $EG = 0$. (2) $BD = 17.5$ kips C; $EG = 55.7$ kips C.

3. In Fig. 128, assume that the knee braces extend from B to I and from J to G instead of as shown. Solve for the stresses in these knee braces if all the horizontal reaction is carried at H.

Ans. $BI = 15.66$ kips C; $JG = 50.0$ kips C.

43. Stresses in Bents: Graphic Solution.—In the graphic solution for the reactions and stresses in a bent, the same free bodies are used as were used in the algebraic solution (Art. 42). The entire bent is used first in order to obtain the reactions at the bases of the columns. The left-hand column is then used to obtain the stress in the knee brace, after which the stresses in the first six members of the truss may be determined. Since each of the next two joints has three unknown forces, either half of the bent must be used as the free body in order to determine the stress in the lower chord.

EXAMPLE

In the bent shown in Fig. 130, horizontal wind pressure acts on both columns. The right-hand column is assumed to take all the horizontal

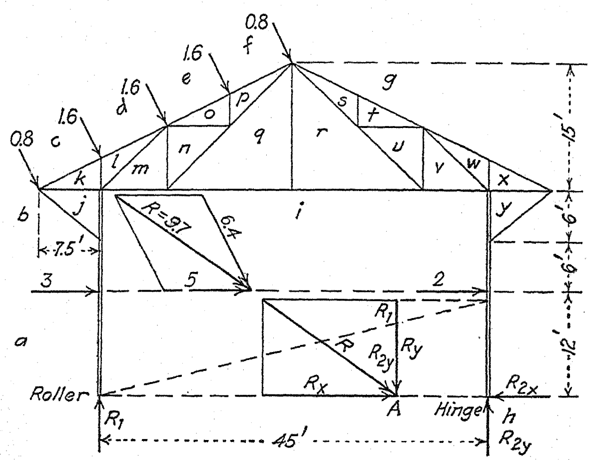


FIG. 130.

reaction. Solve for the reactions, for the stress in the left knee brace, and for the stress in the lower chord qi.

Solution.—With the entire bent as the free body (Fig. 130), the known loads are combined into their resultant $R = 9.7$ kips. This resultant is transmitted along its line of action until it intersects the line between the bases of the columns, at point A. At this point, it is resolved into its vertical and horizontal componets R_y and R_x. The horizontal component R_x scales 7.86 kips and must be balanced by R_{2x}. By the method of inverse

proportion, the vertical component R_y is resolved into two components acting at the columns. The reactions R_1 and R_{2y} must be equal and opposite to these two components. R_1 scales 1.43 kips, and R_{2y} scales 4.29 kips.

Figure 131 shows the solution for the stress in the knee brace on the left-hand column. The reaction R_1 and the 3-kip load are combined into their resultant, 3.32 kips. At the point where this resultant intersects the line of action of the knee brace, at A, the resultant of all the forces at the upper end of the column must also meet. The solution of the force triangle for these forces is BCD, Fig. 131(b). Vector CD is the stress in the knee brace and scales 7.68 kips tension.

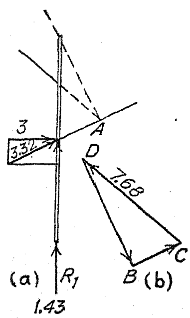


FIG. 131.

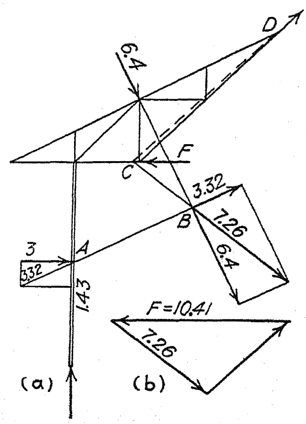


FIG. 132.

In Fig. 132(a), the left half of the bent is shown as a free body in order to solve for the stress F in the lower chord. The 3-kip force and the 1.43-kip reaction are combined at A into their resultant, which scales 3.32 kips. This resultant and the 6.4-kip load on the roof are combined at B into their resultant, which scales 7.26 kips. There are now three forces acting on the free body: this last resultant, the stress F, and the reaction at D. These three forces meet at point C. The solution of the force triangle, Fig. 132(b), gives $F = 10.41$ kips. The stresses in the other members of the truss may now be obtained, including the stress in the knee brace on the right-hand column.

Problems

1. Check the value just obtained for the stress in the lower chord of the bent in the example above by using the right half of the bent as the free body.

2. With the right-hand column of the bent shown in Fig. 130 as the free body, solve for the stress in the knee brace yg. Solve also for the stresses in members iv, vw, wx, and xy.

Ans. $YG = 35.2$ kips T; $IV = 23.56$ kips C; $VW = 37.2$ kips C; $WX = 0$; $XY = 71.5$ kips C.

3. In the bent shown in Fig. 133, the columns are assumed to be perfectly fixed at their lower ends, so that they have points of counterflexure midway between the point of attachment of the knee brace and the lower end. (See discussion in Art. 42.) The horizontal wind load, acting on the left-hand column only, is 0.3 kip per vertical foot of the column. Assuming the

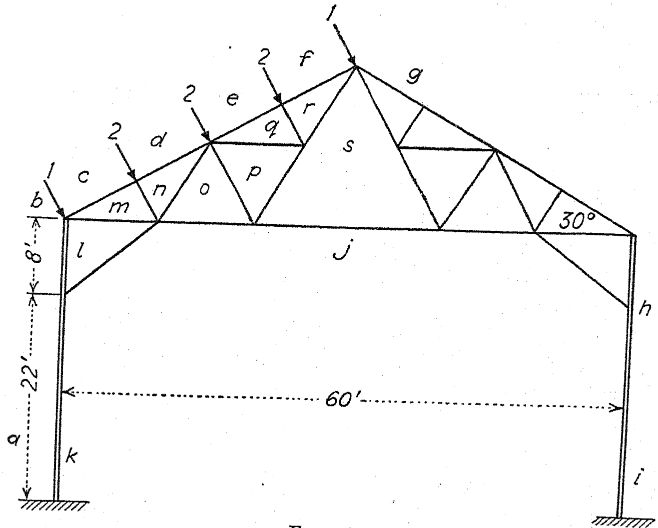


FIG. 133.

horizontal components of the reactions at the points of counterflexure to be equal, solve for the stresses in all the two-force members in the left half of the bent.

Ans. $LJ = 6.08$ kips T; $CM = 10.77$ kips C; $ML = 3.23$ kips T; $SJ = 2.40$ kips C; $DN = 10.77$ kips C; $NM = 2$ kips C; $NO = 6.38$ kips T; $OJ = 3.79$ kips T; $OP = 6.19$ kips C; $PS = 6.19$ kips T; $EQ = 6.97$ kips C; $QP = 2$ kips T; $FR = 6.97$ kips C; $RQ = 2$ kips C; $RS = 8.19$ kips T.

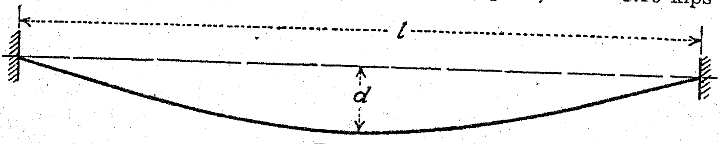


FIG. 134.

44. Flexible Cord: Load Uniform Horizontally.—If a flexible cord is suspended from two points and has a load that is uniformly distributed horizontally, the cord takes the shape of a parabola, as will be shown later. A tightly stretched horizontal wire, rope, chain, or cable with uniform cross section very closely approxi-

mates the condition of uniform loading horizontally. Even with the sag as great as 10 per cent of the span, as shown in Fig. 134, the error in assuming such loading is only about 1.5 per cent.

If the cord is loaded with equal concentrated loads, equally spaced horizontally and with the spaces small compared with the span, the smooth curve through the points of application of the loads takes the shape of a parabola. The cord itself deviates

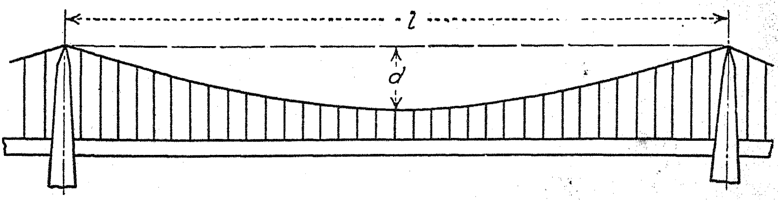


Fig. 135.

only slightly from the parabola. The ordinary suspension bridge with cables, chains, or eyebars as the supporting cords (Fig. 135) illustrates the case of uniformly spaced concentrated loads. The slightly greater weight of the cord and suspenders near the ends of the span can be neglected without appreciable error.

Let Fig. 136(a) represent such a flexible cord which has a span of l and a sag of d, with a load of w per unit horizontal distance.

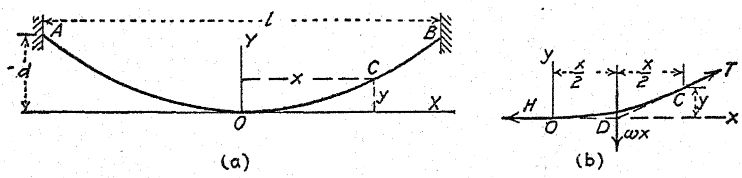


Fig. 136.

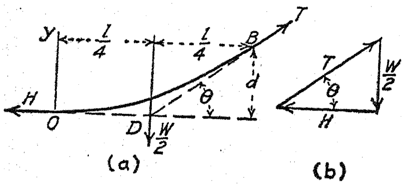
Let the coordinate axes be taken as shown, and let the part OC of the cord be taken as a free body, as shown in Fig. 136(b). The weight supported by this part of the cord is wx, acting at a distance of $x/2$ from C. The three forces on this free body are H, wx, and T, which must be concurrent at D. Since this system of forces is in equilibrium, $\Sigma M_C = 0$.

$$Hy = \frac{wx^2}{2}$$

$$x^2 = \frac{2H}{w}y$$

It will be seen that this is the equation of a parabola with its axis vertical.

In Fig. 137(a), the right half of the cord is taken as a free body. The weight supported by this part of the cord is $wl/2 = W/2$, acting at the distance $l/4$ from B. The tension at point O is H, acting horizontally. The tension at B is T, acting tangent to the curve, at an angle of θ with the horizontal. These three forces meet at point D and constitute a system in equilibrium.



IG. 137.

Figure 137(b) shows the force triangle for the system. From Fig. 137(a),

$$\tan \theta = \frac{4d}{l}$$

From Fig. 137(b),

$$\tan \theta = \frac{W}{2H}$$

If these two values of $\tan \theta$ are equated,

$$\frac{4d}{l} = \frac{W}{2H}$$

$$H = \frac{Wl}{8d}$$

From the relation of the sides of the force triangle,

$$T = \sqrt{\left(\frac{W}{2}\right)^2 + H^2}$$

For a cord stretched with a tension that is large compared with its weight, the value of H approaches T. With the sag 1 per cent of the span, the difference in the values of H and T is only eight-hundredths of 1 per cent. With the sag as large as 5 per cent of the span, the difference is less than 2 per cent. In problems in which the percentage of sag is small, therefore, H may be assumed equal to T with very slight error.

Let s represent the length of the cord between the points of suspension.

$$s = \int ds = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Since $\dfrac{dy}{dx} = \dfrac{wx}{H}$, and $H = \dfrac{wl^2}{8d}$, $\dfrac{dy}{dx} = \dfrac{8d}{l^2}x.$

$$s = 2\int_0^{\frac{l}{2}} \sqrt{1 + \frac{64d^2}{l^4}x^2}\, dx$$

$$s = \frac{l}{2}\sqrt{1 + \frac{16d^2}{l^2}} + \frac{l^2}{8d}\sinh^{-1}\frac{4d}{l}$$

If this is expanded by Maclaurin's series, it becomes a series that is converging if the value of d is less than $l/4$.

$$s = l + \frac{8d^2}{3l} - \frac{32d^4}{5l^3} + \frac{256d^6}{7l^5} - \cdots$$

For the usual values of d and l, the first three terms give sufficient accuracy.

In terms of W and H, the expression for the length becomes

$$s = l + \frac{W^2l}{24H^2} - \frac{W^4l}{640H^4} + \frac{W^6l}{7168H^6} - \cdots$$

If s and H are given to find l, a cubic equation results. Since the last three terms are small, however, l in them may be replaced by s with very little error.

$$l = s - \frac{W^2s}{24H^2} + \frac{W^4s}{640H^4} - \frac{W^6s}{7168H^6} + \cdots \text{ (approx.)}$$

If the supports at the ends of the cord are not on the same level, as A and B, Fig. 138, it is necessary to use the two free bodies AC and CB and to write two moment equations $\Sigma M_A = 0$ and $\Sigma M_B = 0$.

$$Hd_1 = \frac{wl_1^2}{2}$$

$$Hd_2 = \frac{wl_2^2}{2}$$

$$\frac{d_1}{d_2} = \frac{l_1^2}{l_2^2}$$

If l_2 is replaced by $l - l_1$, the last equation when solved for l_1 becomes

$$l_1 = \frac{\sqrt{d_1 d_2} - d_1}{d_2 - d_1} l$$

If the cord is supported so that the vertex of the parabola is not between the supports, as BD, Fig. 138, the curve must be extended down to the vertex. The horizontal distance between points C and D then becomes l_1, and the horizontal distance

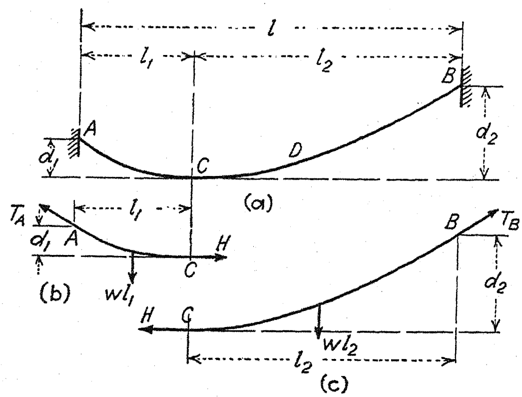


FIG. 138.

between C and B becomes l_2. In the equation for l_1, the length l must be replaced by the given horizontal span plus $2l_1$.

EXAMPLE 1

A cable has a span of 800 ft. and a sag of 50 ft. and supports a load of 600 lb. per horizontal foot of cable. Compute the tension at the bottom point of the cable, the tension at the ends of the cable, and the total length of the cable between supports.

Solution.—In Fig. 137(a), $l/4 = 200$ ft., $d = 50$ ft., and $W/2 = 400 \times 600 = 240{,}000$ lb. The equation of moments with respect to point B gives

$$50H - 240{,}000 \times 200 = 0$$
$$H = 960{,}000 \text{ lb.}$$
$$T = \sqrt{960{,}000^2 + 240{,}000^2} = 989{,}600 \text{ lb.}$$
$$s = 800 + \frac{8 \times 50^2}{3 \times 800} - \frac{32 \times 50^4}{5 \times 800^3} + \frac{256 \times 50^6}{7 \times 800^5}$$
$$s = 800 + 8.333 - 0.078 + 0.002 = 808.257 \text{ ft.}$$

It will be noticed that in this problem the first two terms give sufficient accuracy for all practical purposes.

EXAMPLE 2

A steel wire 0.12 in. in diameter is to be stretched between two points on the same level 100 ft. apart. If the allowable stress in the wire is 12,000 lb./sq. in., what will be the sag and the length of the wire? Steel weighs 490 lb./cu. ft.

Solution.—The cross section of the wire is 0.0113 sq. in. The weight of one-half the wire is

$$\frac{W}{2} = 0.0113 \times 50 \times \frac{490}{144} = 1.923 \text{ lb.}$$

The allowable tension in the wire is

$$T = 0.0113 \times 12{,}000 = 135.6 \text{ lb.}$$

The free-body diagram of the right half of the wire is shown in Fig. 139. The equation of moments with respect to point O gives

$$135.6 \times 25 \sin \theta - 25 \times 1.923 = 0$$
$$\sin \theta = 0.01418$$

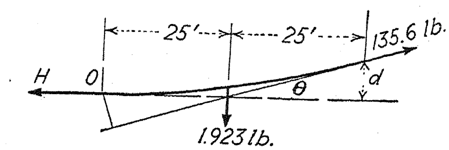


FIG. 139.

For an angle as small as this, the tangent may be considered as equal to the sine, so $\tan \theta = 0.01418$.

$$d = 25 \times 0.01418 = 0.3545 \text{ ft.}$$

For a wire with as small a deflection as this, the third and fourth terms in the expression for s may be neglected.

$$s = 100 + \frac{8 \times 0.3545^2}{3 \times 100} = 100.003 \text{ ft.}$$

EXAMPLE 3

A cable weighing 2 lb./lin. ft. is suspended between points 600 ft. apart horizontally and 20 ft. apart vertically. The cable sags 10 ft. below the level of the lower support. Compute the tension at the ends of the cable and the total length of the cable.

Solution.

$$d_1 = 10,\ d_2 = 30,\ w = 2, \text{ and } l = 600.$$
$$l_1 = \frac{\sqrt{300} - 10}{20} \times 600 = 219.6 \text{ ft.}$$
$$l_2 = 380.4 \text{ ft.}$$

$$H = \frac{2 \times 219.6^2}{2 \times 10} = 4822 \text{ lb.}$$
$$wl_1 = 439 \text{ lb.}$$
$$T_A = \sqrt{4822^2 + 439^2} = 4842 \text{ lb.}$$
$$wl_2 = 761 \text{ lb.}$$
$$T_B = \sqrt{4822^2 + 761^2} = 4886 \text{ lb.}$$

In computing s, three terms will be sufficient, since the sag is small.

$$s_1 = 219.6 + \frac{8 \times 100}{3 \times 219.6} - \frac{32 \times 10{,}000}{5 \times 219.6^3}$$
$$s_1 = 219.6 + 1.214 - 0.006 = 220.808 \text{ ft.}$$
$$s_2 = 380.4 + \frac{8 \times 900}{3 \times 380.4} - \frac{32 \times 810{,}000}{5 \times 380.4^3}$$
$$s_2 = 380.4 + 6.309 - 0.094 = 386.615 \text{ ft.}$$
$$s = 607.423 \text{ ft.}$$

Problems

1. A cable having a span of 240 ft. and a sag of 32 ft. supports a load of 500 lb. per horizontal foot. Solve for the maximum stress and the length of the cable. *Ans.* 127,500 lb.; 250.942 ft.

2. Copper transmission wire ¼ in. in diameter has a maximum allowable stress of 6000 lb./sq. in. If coated with ice until it has a diameter of 1 in., what should be the spacing of poles if the sag is not to exceed 4 ft.? Copper weighs 556 lb./cu. ft. Ice weighs 55 lb./cu. ft. *Ans.* 141 ft.

3. The bridge across the Golden Gate at San Francisco has a span of 4200 ft. and a sag of 475 ft., and each cable carries a load of 12,500 lb. per horizontal foot. Compute the maximum tension in the cable and the length of the cable between towers. Using only the first three terms of the expression for s, compute the change in sag for a change of 50°F. in the cable. Use $\epsilon = 0.0000067$. *Ans.* 63,700,000 lb.; 4339.0 ft.; 2.31 ft.

4. A cable weighing 5 lb./ft. is stretched between points that are 500 ft. apart horizontally and 40 ft. apart vertically. The cable sags 8 ft. below the lower support. Compute the maximum tension in the cable and its length. *Ans.* 6805 lb.; 517.815 ft.

5. The cable of a suspension bridge between the end tower and the anchorage carries a load of 1500 lb. per horizontal foot. The anchorage is 80 ft. from the base of the tower and 60 ft. below the top of the tower. The vertex of the curve would be 5 ft. below the anchorage. Compute the tension in the cable at the anchorage and at the top of the tower. *Ans.* 148,300 lb.; 217,800 lb.

45. Flexible Cord: Load Uniform along Cord.—When the load on a cord is uniformly distributed along its length, the curve that the cord assumes is called the *catenary*. The equation of the catenary curve will now be derived.

Let w be the weight of the cord per unit length. Let O, Fig. 140, be the lowest point on the cord; A any other point; s the

length of the cord from O to A; H the tension at O; and T the tension at A. Also let H be represented by the weight of an imaginary length of the cord c, or $H = wc$. In Fig. 140(a), the length of cord s is shown as a free body in equilibrium. In Fig. 140(b) is shown the force triangle, from which is obtained the relation

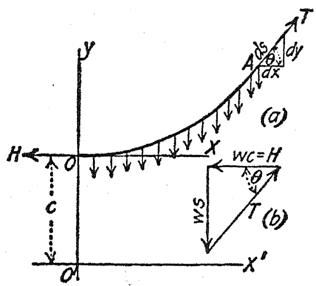


FIG. 140.

$$\frac{ws}{wc} = \tan\theta$$

or

$$\frac{dy}{dx} = \frac{s}{c}$$

$$dy^2 = ds^2 - dx^2$$

so

$$\frac{s}{c} = \frac{\sqrt{ds^2 - dx^2}}{dx}$$

By squaring and solving for dx,

$$dx = \frac{c\,ds}{\sqrt{c^2 + s^2}}$$

or

$$\int_0^x dx = c\int_0^s \frac{ds}{\sqrt{c^2 + s^2}}$$

By integration,

$$x = c \log_e \frac{s + \sqrt{c^2 + s^2}}{c} \tag{1}$$

The quantity e is the base of the Napierian system of logarithms, and its numerical value is 2.71828. The reduction to common logarithms is made by the relation

$$0.4343 \log_e A = \log_{10} A$$

Reduced to exponential form, equation (1) becomes

$$e^{\frac{x}{c}} = \frac{s + \sqrt{c^2 + s^2}}{c}$$

Solution for s gives

$$s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) \tag{2}$$

If this value of s is substituted in the original equation, the expression

$$dy = \tfrac{1}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}})dx$$

is obtained.

If the origin is at O, and dy is integrated between the limits 0 and y, a complicated expression results. A simpler expression is obtained by using O' as the origin. The integration of dy is then between the limits c and y.

$$\int_0^y dy = \frac{c}{2}\left(\int_0^x \frac{1}{c}e^{\frac{x}{c}}\,dx - \int_c^x \frac{1}{c}e^{-\frac{x}{c}}\,dx\right)$$

By integration,

$$y - c = \frac{c}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}}) - c$$

or

$$y = \frac{c}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}}) \qquad (3)$$

By squaring (2) and (3) and subtracting,

$$y^2 = s^2 + c^2 \qquad (4)$$

From (1) and (4),

$$x = c \log_e \frac{s + y}{c} \qquad (5)$$

From the relation of the sides of the force triangle, Fig. 140(b),

$$T^2 = w^2c^2 + w^2s^2 = w^2(s^2 + c^2) = w^2y^2$$

$$T = wy \qquad (6)$$

The related quantities are as follows:

Length	Unit Weight	Tension	Span	Deflection
$2s$	w	T	$2x_1 = l$	$y - c$

The most useful problems, those in which the *length, span,* and *weight,* or the *deflection, span,* and *weight* are given, can be solved for the unknown quantities only by trial, on account of the form of the logarithmic equation.

If the supports at the ends of the cord are not on the same level, it is necessary to solve the two parts of the curve separately. If

x_1 and x_2 are the horizontal distances from the vertex of the curve to the vertical lines through the supports, the span $l = x_1 + x_2$. The value of c is the same for both parts of the curve. The length of the curve is $s = s_1 + s_2$.

EXAMPLE 1

A cable 800 ft. long, weighing 2 lb./ft., is stretched between two points on the same level with a tension of 1200 lb. Compare the sag and the span.

Solution.

$$w = 2 \text{ and } T = 1200$$

Equation (6) gives

$$y = 600 \text{ ft.}$$

From equation (4),

$$c = 447.2 \text{ ft.}$$

The sag is $y - c = 152.8$ ft.

From equation (5) the span is $2x = 2 \times 447.2 \log_e \dfrac{1000}{447.2}$

$$2x = 719.8 \text{ ft.}$$

EXAMPLE 2

If a cable weighing 2 lb./ft. is stretched between points 800 ft. apart and sags 100 ft., what tension and length of cable are required?

Solution.—This problem can be solved only by trial.

$$y = c + 100$$

From equation (4),

$$c^2 + 200c + 10{,}000 = s^2 + c^2$$
$$s = 10\sqrt{2c + 100}$$

From equation (5),

$$400 = c \log_e \frac{10\sqrt{2c + 100} + c + 100}{c}$$

It is found by trial that $c = 810$ will nearly satisfy this equation. With this value of c,

$$y = 910 \text{ and } T = 1820$$

From equation (4),

$$s^2 = y^2 - c^2$$
$$s = 414.7 \text{ ft.}$$

Total length $= 2s = 829.5$ ft.

EXAMPLE 3

A cable weighing 4 lb./lin. ft. is suspended from points 800 ft. apart horizontally and 100 ft. apart vertically. Compute the tension at each end and the length of the cable if it sags 200 ft. below the lower support.

Solution.—This problem can be solved only by trial.

$$y_1 = c + 200;\ y_2 = c + 300$$

From equation (4),

$$s_1 = 20\sqrt{c + 100}, \text{ and } s_2 = 20\sqrt{1.5c + 225}$$

From equation (5),

$$800 = c\left(\log_e \frac{20\sqrt{c + 100} + c + 200}{c} + \log_e \frac{20\sqrt{1.5c + 225} + c + 300}{c}\right)$$

By trial it is found that $c = 360$ will very nearly satisfy this equation. With this value of c,

$$\begin{aligned} y_1 &= 560 \text{ ft., and } T_1 = 2240 \text{ lb.} \\ y_2 &= 660 \text{ ft., and } T_2 = 2640 \text{ lb.} \\ s_1 &= \sqrt{560^2 - 360^2} = 429 \text{ ft.} \\ s_2 &= \sqrt{660^2 - 360^2} = 553 \text{ ft.} \\ s &= 429 + 553 = 982 \text{ ft.} \end{aligned}$$

Problems

1. A wire 500 ft. long weighing 0.094 lb./lin. ft. has a tension of 30 lb. at each end. Compute the span and the sag. *Ans.* 418 ft.; 121 ft.

2. A chain 100 ft. long weighing 2 lb./lin. ft. is suspended between two points on the same level 60 ft. apart. Compute the tension at the ends and the sag. *Ans.* 105.1 lb.; 36.35 ft.

3. A cable weighing 1.6 lb./lin. ft. carries sheathed telephone cable and supports weighing 1.2 lb./lin. ft. The span between towers is 600 ft., and the sag is 75 ft. Compute the length of the cable and the maximum tension, assuming the supporting cable to carry all the stress.

Ans. 622 ft.; 1910 lb.

4. A cable weighing 1 lb./lin. ft. is suspended between points 300 ft. apart horizontally and 40 ft. apart vertically. The cable sags 20 ft. below the lower support. Compute the tension at each end and the length of the cable. *Ans.* 328 lb.; 368 lb.; 314 ft.

GENERAL PROBLEMS ON COPLANAR, NONCONCURRENT FORCES

1. Solve for the stresses in members AB, AC, and BC of the truss[1] shown in Fig. 141.

Ans. $AB = 11.2$ kips C; $AC = 11.6$ kips T; $BC = 6$ kips C.

[1] Consider all trusses in this set of problems to be pin-connected at all joints.

2. Solve for the stresses in the members of the truss shown in Fig. 142.
Ans. $AB = 9$ kips C; $AD = 7.79$ kips T; $BD = 8$ kips T.

3. On the truss shown in Fig. 143, the reaction at B is horizontal; that at A is both vertical and horizontal. Solve for these components of the reactions and for the stresses in the members of the truss.

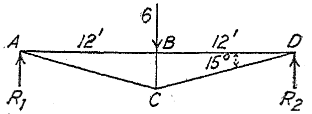


FIG. 141.

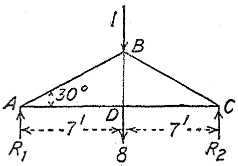


FIG. 142.

Ans. $B = 6.928$ kips; $A_x = 6.928$ kips; $A_y = 7$ kips; $AB = 4$ kips T; $AC = 6$ kips T; $AD = 1.732$ kips T; $BC = 8$ kips C; $DC = 2$ kips C; $CE = 2$ kips C; $DE = 1.732$ kips T.

4. In Fig. 143, consider the 4-kip force to act horizontally to the right, the other forces remaining as shown. Solve for the reactions and for the stresses that are changed.

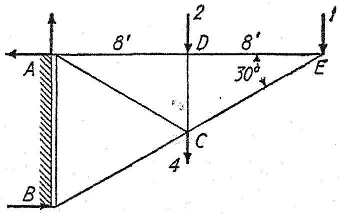


FIG. 143.

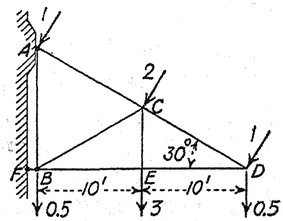


FIG. 144.

Ans. $B = 1.46$ kips; $A_x = 5.46$ kips; $A_y = 3$ kips; $AB = 0.845$ kips T; $AC = 4.31$ kips T; $BC = 1.69$ kips C.

5. In the truss shown in Fig. 144, the reaction of the short strut FB is horizontal. Solve for the pressure FB, the vertical and horizontal components of the reaction at A, and all the internal stresses.

Ans. $FB = 7.46$ kips C; $A_x = 5.46$ kips; $A_y = 7.46$ kips; $AB = 3.15$ kips T; $AC = 6.89$ kips T; $BC = 5.31$ kips C; $BE = ED = 2.87$ kips C; $CE = 3$ kips T; $CD = 2.732$ kips T.

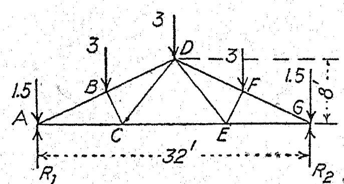


FIG. 145.

6. Solve for all the internal stresses in the truss shown in Fig. 145.

Ans. $AB = 10.06$ kips C; $AC = 9$ kips T; $BC = 2.68$ kips C; $BD = 8.72$ kips C; $CD = 3$ kips T; $CE = 6$ kips T.

7. Determine by inspection which of the members of the truss shown in Fig. 145 would have their stresses changed by a load at E. Solve for the

amount of additional stress in these members caused by a load of 4 kips

Ans. $AB = BD = 2.8$ kips C; $AC = CE = 2.5$ kips T; $DE = 5$ kips T; $DF = FG = 6.15$ kips C; $EG = 5.5$ kips T.

8. Solve for all the stresses in the members of the cantilever truss shown in Fig. 146.

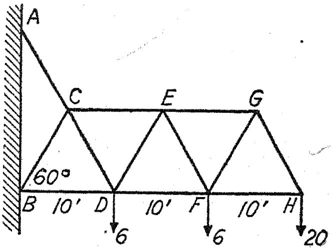


Fig. 146.

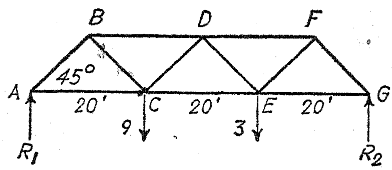


Fig. 147.

Ans. $AC = 90.1$ kips T; $BC = 53.1$ kips T; $BD = 71.6$ kips C; $CD = 37$ kips T; $CE = 53.1$ kips T; $DE = 30$ kips C; $DF = 38.1$ kips C; $EF = 30$ kips T; $EG = 23.1$ kips T; $FG = 23.1$ kips C; $FH = 11.55$ kips C; $GH = 23.1$ kips T.

9. Solve for all the internal stresses in the truss shown in Fig. 147.

Ans. $AB = 9.9$ kips C; $AC = 7$ kips T; $BD = 14$ kips C; $BC = 9.9$ kips T; $CD = 2.83$ kips T; $CE = 12$ kips T; $DE = 2.83$ kips C; $DF = 10$ kips C; $EF = 7.07$ kips T; $EG = 5$ kips T; $FG = 7.07$ kips C.

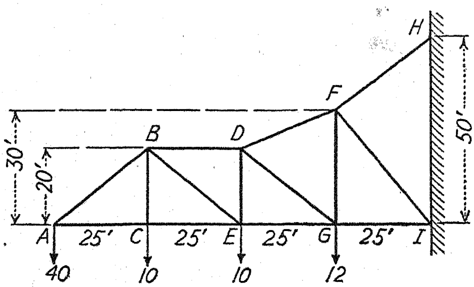


Fig. 148.

10. Solve for the amount and direction of the reaction at I on the truss shown in Fig. 148 and for the stresses in all the members.

Ans. $I = 112$ kips, 8°35′ with X; $AB = 64$ kips T; $AC = CE = 50$ kips C; $BC = 10$ kips T; $BD = 112.5$ kips T; $BE = 80$ kips C; $DE = 60$ kips T; $EG = 112.5$ kips C; $DF = 134.5$ kips T; $DG = 16$ kips C; $FG = 22$ kips T; $GI = 125$ kips C; $FH = 142$ kips T; $FI = 21.9$ kips T.

11. In the truss shown in Fig. 149, solve for the stresses in members AB, AC, BD, and BE. Assuming that members DG and EF can take only tension, determine which one is acting and the stress in it.

Ans. $AB = 26.12$ kips C; $AC = 16.32$ kips T; $BD = 23.04$ kips C; $BE = 10.76$ kips T; $EF = 4.61$ kips T.

12. Solve for all the stresses in the members of the truss shown in Fig. 150. The four diagonals DG, EF, FI, and GH can take tension only.

Ans. AB = 48.5 kips C; $AC = CE$ = 31.1 kips T; BC = 20 kips T; BE = 12.42 kips T; BD = 39.82 kips C; DE = 6.45 kips T; DF = 40.05 kips

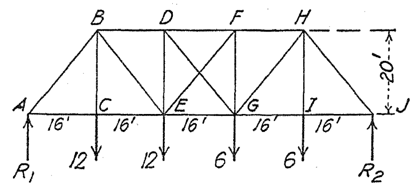


FIG. 149.

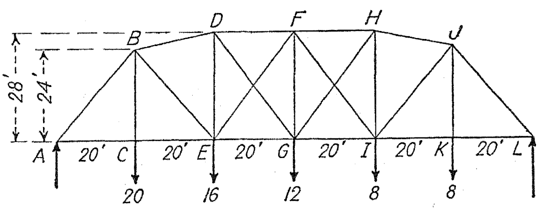


FIG. 150.

C; EG = 39.06 kips T; DG = 1.64 kips T; FG = 0; FH = 40.05 kips C; GH = 13.1 kips T; GI = 32.4 kips T; HI = 4.2 kips C; HJ = 33.1 kips C; IJ = 15.9 kips T; $IK = KL$ = 22.22 kips T; JK = 8 kips T; JL = 34.71 kips C.

13. Solve for the stresses in members a, b, c, and d in the truss shown in Fig. 151.

Ans. a = 15 kips C; b = 8.66 kips T; c = 8.66 kips T; d = 1.732 kips T.

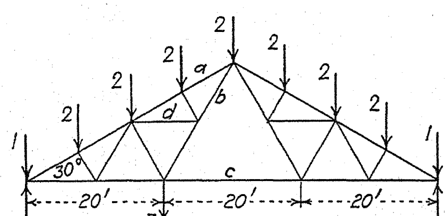


FIG. 151.

14. Solve for the reactions and for all the stresses in the roof truss shown in Fig. 152.

Ans. R_1 = 13.58 kips; R_2 = 15.85 kips; R_3 = 2.97 kips; AB = 34.06 kips C; $AC = CE$ = 31.25 kips T; BC = 4 kips T; BD = 26.59 kips C; BE = 8.27 kips C; DE = 7.08 kips T; DF = 19.05 kips C; DG = 9.98 kips C; EG = 23.66 kips T; FG = 12.23 kips T; FH = 10.3 kips C; FI =

14.10 kips C; GI = 15.88 kips T; HI = 18.74 kips T; HJ = 18.70 kips C; IJ = 6.93 kips T.

15. In Fig. 152, solve for the stresses in DF, DG, and EG due to the loads on the lower chord alone, without getting the stresses in any of the other members. *Ans.* DF = 15.05 kips C; DG = 6.4 kips C; EG = 19 kips T.

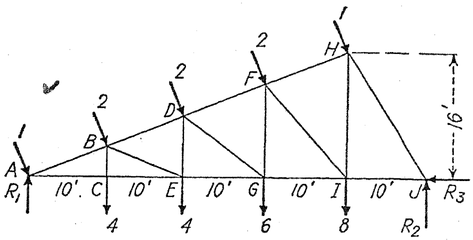


FIG. 152.

16. In the bent shown in Fig. 153, assume that reaction R_1 is vertical. Solve for the stresses in members a, b, c, d, e, f, and g.

Ans. a = 2.82 kips T; b = 1.30 kips T; c = 10.22 kips C; d = 11.55 kips C; e = 17.7 kips T; f = 4.80 kips C; g = 22.7 kips C.

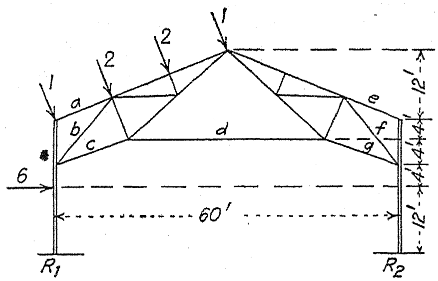


FIG. 153.

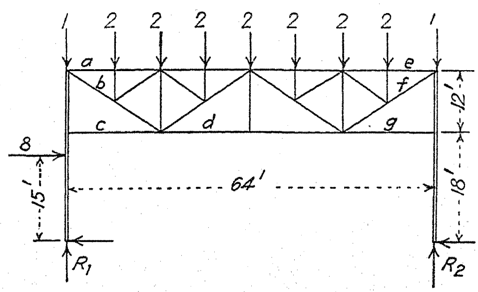


FIG. 154.

17. In the bent shown in Fig. 154, assume the horizontal components of the reactions to be equal. Solve for the stresses in members a, b, c, d, e, f, and g.

Ans. a = 10.83 kips C; b = 8.54 kips T; c = 0; d = 5.67 kips T; e = 5.83 kips C; f = 14.79 kips T; g = 10 kips C.

18. In the bent shown in Fig. 155, the right-hand column is assumed to take all the horizontal reaction. Solve for the stresses in members a, b, c, d, e, f, and g.

Ans. a = 4.03 kips C; b = 3 kips C; c = 14.1 kips T; d = 7.36 kips C; e = 19.44 kips C; f = 24.9 kips C; g = 60.5 kips C.

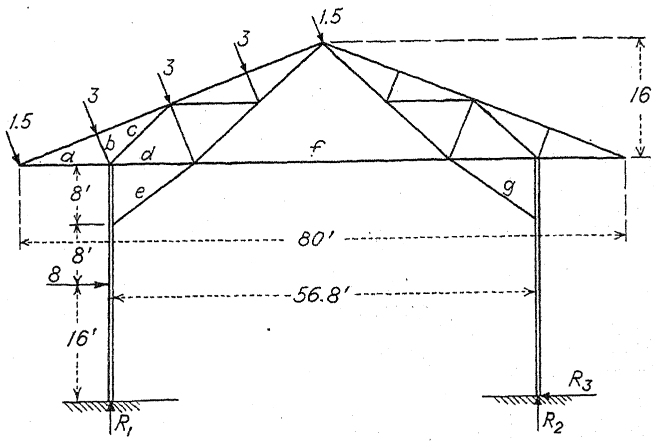


FIG. 155.

19. Figure 156 represents a wheel weighing 600 lb. supported on an axle at O and carrying a load of 1800 lb. suspended from a cable wrapped around the wheel. Solve for the force P to balance the load. Solve also for R_x and R_y. *Ans.* P = 3720 lb.; R_x = 3600 lb.; R_y = 3365 lb.

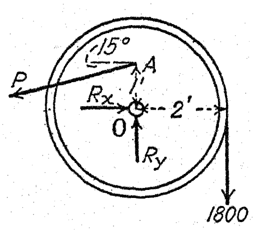


FIG. 156.

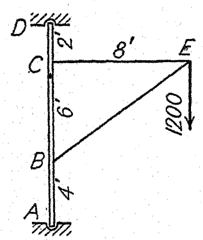


FIG. 157.

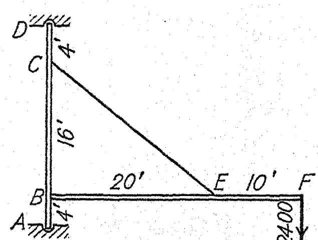


FIG. 158.

20. The 1200-lb. pull on the crane shown in Fig. 157 may be rotated around point E in a vertical plane through CE. Locate the position of the pull to give the maximum tension in CE. Solve for this maximum tension, for the stress in BE, and for the reactions at A and D.

Ans. 36°52′ with the vertical; CE = 2000 lb.; BE = 1600 lb.; A = 1090 lb., 61°30′ with X; D = 1240 lb.

21. In the crane shown in Fig. 158, solve for the vertical and horizontal components of the reactions at A and B, for the reaction at D, and for the stress in CE due to the load of 2400 lb.

Ans. $A_x = 3000$ lb.; $A_y = 2400$ lb.; $D = 3000$ lb.; $B_x = 4500$ lb.; $B_y = 1200$ lb.; $CE = 5760$ lb. T.

22. Solve Prob. 21 if the boom and the mast each weigh 40 lb./lin. ft.

Ans. $A_x = 3750$ lb.; $A_y = 4560$ lb.; $D = 3750$ lb.; $B_x = 5625$ lb.; $B_y = 900$ lb.; $CE = 7200$ lb. T.

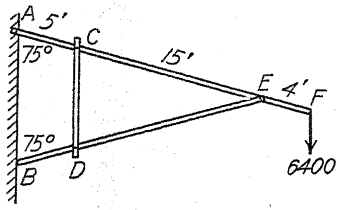


Fig. 159.

23. The frame shown in Fig. 159 is pinned at A, C, D, and E, and is supported by a smooth vertical wall at B. Solve for the vertical and horizontal components of the reactions at A and E, the stress in CD, and the reaction at B due to the 6400-lb. load.

Ans. $A_x = 14{,}340$ lb.; $A_y = 6400$ lb.; $E_x = 14{,}340$ lb.; $E_y = 5130$ lb.; $CD = 5130$ lb.; $B = 14{,}340$ lb.

24. In Fig. 159, consider the members to weigh 30 lb./lin. ft. Solve for the vertical and horizontal components of the reactions at all points.

Ans. $A_x = 15{,}800$ lb.; $A_y = 7953$ lb.; $B = 15{,}800$ lb.; $C_x = 0$; $C_y = 6280$ lb.; $D_x = 0$; $D_y = 6050$ lb.; $E_x = 15{,}800$ lb.; $E_y = 5450$ lb.

25. The camp stool shown in Fig. 160 rests on a smooth floor at D and E. Solve for the reactions at D and E and for the vertical and horizontal components of the pin pressures at A, B, and C.

Ans. $D = 96$ lb.; $E = 64$ lb.; $A_x = 206$ lb.; $A_y = 102.9$ lb.; $B_x = 206$ lb.; $B_y = 57.1$ lb.; $C_x = 206$ lb.; $C_y = 38.9$ lb.

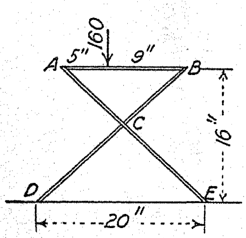


Fig. 160.

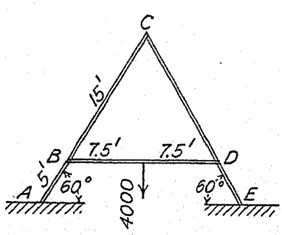


Fig. 161.

26. The A-frame shown in Fig. 161 is pinned at E and supported by a smooth floor at A. Solve for the reactions at A and E and for the vertical and horizontal components of the pin pressures at B, C, and D due to the 4000-lb. load.

Ans. $A = E = 2000$ lb.; $B_y = D_y = 2000$ lb.; $B_x = C_x = D_x = 385$ lb.; $C_y = 0$.

27. Solve for the reactions and pin pressures on the A-frame shown in Fig. 161 if the members weigh 40 lb./lin. ft.

Ans. $A = E = 3100$ lb.; $B_y = D_y = 2300$ lb.; $B_x = C_x = D_x = 750$ lb.; $C_y = 0$.

28. Solve for the reactions and pin pressures on the A-frame shown in Fig. 161 due to the 4000-lb. load alone if it is moved to a point 2 ft. from D.

Ans. $A = 900$ lb.; $E = 3100$ lb.; $B_y = 533$ lb.; $D_y = 3467$ lb.; $C_y = 367$ lb.; $B_x = C_x = D_x = 385$ lb.

29. A suspension footbridge is 120 ft. long and 4 ft. wide and carries a load of 160 lb./sq. ft. of floor area. It is supported by two cables which have a sag of 15 ft. Compute the maximum stress in each cable and the length of each cable between supports. *Ans.* 42,900 lb.; 124.82 ft.

30. A wire can safely sustain 250 lb. tension. Its weight per linear foot is 0.06 lb. Compute the maximum spacing for posts if the allowable sag is 3 in. (Assume $H = T$.) *Ans.* 91 ft.

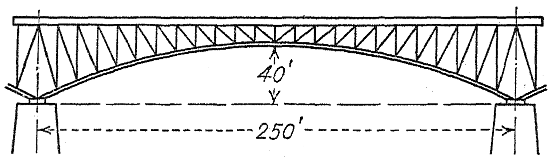


FIG. 162.

31. The bridge shown in Fig. 162 has its lower chord in the shape of a parabola. The weight carried by one truss is 2400 kips. Compute the stress in the lower chord at the middle and at the piers. (This bridge is like the suspension bridge inverted.) *Ans.* 1875 kips; 2226 kips.

32. A chain 60 ft. long weighing 1.2 lb./lin. ft. has a tension of 40 lb. at each end. Compute the span and the sag.

Ans. Span, 42.8 ft.; sag, 18.8 ft.

33. A transmission cable 1200 ft. long weighing 1.5 lb./lin. ft. is suspended between points on the same level 1180 ft. apart. Compute the sag and the tension at the ends. *Ans.* 103 ft.; 2704 lb. ($c = 1700$ ft.)

34. A cable 800 ft. long, weighing 1.5 lb./lin. ft., has a tension of 750 lb. at each end. Compute the sag and the distance between supports.

Ans. 200 ft.; 660 ft. ($c = 300$ ft.)

CHAPTER V

CONCURRENT FORCES IN SPACE

46. Resolution of a Force into Three Rectangular Components. The most common case of the resolution of a force into three components is that in which the components are parallel, respectively, to the three rectangular axes.

If the angles between the force and the three rectangular axes are given, α with X, β with Y, and γ with Z, it is plain from the trigonometry of Fig. 163 that the rectangular components along the three axes are given by

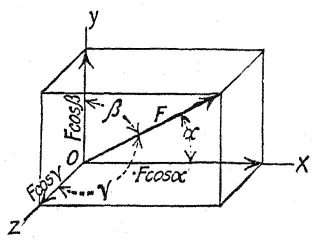


Fig. 163.

$$F_x = F \cos \alpha, F_y = F \cos \beta, \text{ and } F_z = F \cos \gamma$$

If it is desired to make the resolution graphically, a plane may be passed through the force and each axis in turn. In this plane, the angle is shown in its true size, and the projection of the force vector upon the axis gives the amount of the component in that direction.

If the rectangular coordinates of two points on the line of action of the force are given instead of the angles α, β, and γ, their cosines may be computed from the dimensions given, and the resolution may be made as above.

In the graphic solution, a plane is passed through the force and one of the axes, say the Y axis, and the force is resolved into two components, one along the Y axis F_y, and the other in the XZ plane. In this XZ plane the latter component is resolved into F_x and F_z.

EXAMPLE

Figure 164 represents a force of 100 lb. acting along the diagonal of a parallelepiped, the dimensions of which in the X, Y, and Z directions are 12, 6, and 8 ft., respectively. Resolve the force into its three rectangular components.

Solution.—The length of the diagonal OA is given by the square root of the sum of the squares of the three sides.

$$OA = \sqrt{244} = 15.62 \text{ ft.}$$

$$\text{Angle } \alpha \text{ is angle } AOB;\ \cos AOB = \frac{12}{15.62}$$

$$F_x = 100 \times \frac{12}{15.62} = 76.8 \text{ lb.}$$

$$\text{Angle } \beta \text{ is angle } AOC;\ \cos AOC = \frac{6}{15.62}$$

$$F_y = 100 \times \frac{6}{15.62} = 38.4 \text{ lb.}$$

$$\text{Angle } \gamma \text{ is angle } AOD;\ \cos AOD = \frac{8}{15.62}$$

$$F_z = 100 \times \frac{8}{15.62} = 51.2 \text{ lb.}$$

If the force had been acting in the opposite direction, all the components would have been the same in amount but negative in sign. If it had been

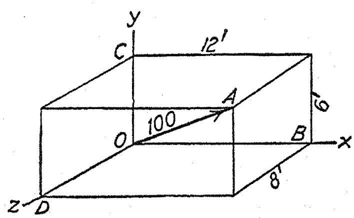


FIG. 164.

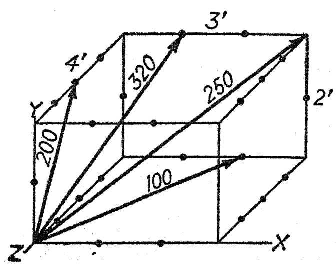


FIG. 165.

acting along any of the other diagonals of the parallelepiped, some of the components would have been positive, and some negative.

Problems

1. In Fig. 165, resolve the 250-lb. force into its three rectangular components. *Ans.* $F_x = 139.3$ lb.; $F_y = 92.8$ lb.; $F_z = -185.7$ lb.

2. In Fig. 165, resolve the 320-lb. force into its three rectangular components. Compute also the angles α, β, and γ.

Ans. $F_x = 69.8$ lb.; $F_y = 139.7$ lb.; $F_z = -279.3$ lb.; $\alpha = 77°25'$; $\beta = 64°10'$; $\gamma = 150°50'$.

47. Resultant of Concurrent Forces in Space.—If a system of noncoplanar concurrent forces is to be combined into their resultant, the method of resolution and recomposition is the simplest. At the common point of intersection, each force may be resolved into its X, Y, and Z components. The X components may be added algebraically into ΣF_x, the Y components

into ΣF_y, and the Z components into ΣF_z. The final resultant of these three is

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$$

The angles that R makes with the three axes are given by the following expressions:

$$\alpha = \cos^{-1} \frac{\Sigma F_x}{R};\ \beta = \cos^{-1} \frac{\Sigma F_y}{R};\ \gamma = \cos^{-1} \frac{\Sigma F_z}{R}$$

It is obvious that the graphic method is also easily applicable to this problem.

EXAMPLE

In Fig. 165, combine the 100- and the 250-lb. forces into their resultant.

Solution.—The length of the diagonal along which the 100-lb. force acts is $\sqrt{2^2 + 4^2} = 4.472$ ft. For the 100-lb. force, the X, Y, and Z components are as follows:

$$F_x = \frac{2}{4.472} \times 100 = 44.7 \text{ lb.};\quad F_y = 0;\quad F_z = -\frac{4}{4.472} \times 100 = -89.4 \text{ lb.}$$

From the answer to Prob. 1 (Art. 46), the three components of the 250-lb. force are obtained.

$$F_x = 139.3 \text{ lb.};\ F_y = 92.8 \text{ lb.};\ F_z = -185.7 \text{ lb.}$$

By adding these, the X, Y, and Z components of the resultant are obtained.

$$\Sigma F_x = 184.0 \text{ lb.};\ \Sigma F_y = 92.8 \text{ lb.};\ \Sigma F_z = -275.1 \text{ lb.}$$

$$R = \sqrt{184^2 + 92.8^2 + 275.1^2} = 343.7 \text{ lb.}$$

$$\cos\alpha = \frac{184.0}{343.7} = 0.536 \qquad \alpha = 57°35'$$

$$\cos\beta = \frac{92.8}{343.7} = 0.27 \qquad \beta = 74°20'$$

$$\cos\gamma = -\frac{275.1}{343.7} = -0.80 \quad \gamma = 143°10'$$

Problems

1. In Fig. 165, combine the 200- and the 320-lb. forces into their resultant. *Ans.* $R = 510.8$ lb.; $\alpha = 82°10'$; $\beta = 56°40'$; $\gamma = 145°30'$.

2. In Fig. 165, reverse the directions of the 100- and the 320-lb. forces, and solve for the resultant of the four forces.

Ans. $R = 106.2$ lb.; $\alpha = 76°30'$; $\beta = 27°10'$; $\gamma = 66°55'$.

48. Moment of a Force with Respect to a Line.—The moment of a force with respect to a line parallel to it is zero, since there is no tendency for the force to rotate the body upon which it acts

about that line as an axis. The moment of a force with respect to an axis intersecting it is zero, since the moment arm is zero. The moment of a force with respect to an axis in a plane perpendicular to the force is equal to the product of the force and the perpendicular distance from the force to the axis.

If the axis is not in a plane perpendicular to the force, the force may be resolved at any point into two rectangular components, one parallel to the axis, the other perpendicular to a plane containing the axis. The moment of the original force with respect to the axis is equal to the moment of the perpendicular component alone, since the moment of the component parallel to the axis is zero.

Another method is to resolve the force into three mutually rectangular components, one of which is parallel to the axis and hence has no moment with respect to it. The sum of the moments of the other two components gives the moment of the original force.

In determining the sign of the moment of a force with respect to one of the coordinate axes, the force is viewed from the positive end of the axis. Moments in a counterclockwise direction when viewed thus are considered positive, and those in a clockwise direction are considered negative.

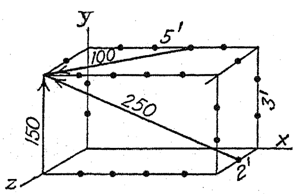


FIG. 166.

Problems

1. In Fig. 166, compute the moment of the 250-lb. force with respect to the lower front edge of the parallelepiped.

Ans. 126.9 lb.-ft.

2. In Fig. 166, compute the resultant moment of the three forces with respect to the X, Y, and Z axes.

Ans. $M_x = -261$ lb.-ft.; $M_y = -589$ lb.-ft.; $M_z = +884$ lb.-ft.

49. Principle of Moments for Concurrent Forces in Space.—It has already been shown in Art. 18 that the sum of the moments of two concurrent forces with respect to a point in their plane is equal to the moment of their resultant with respect to the same point. It will be shown that the same principle holds true not only with respect to any point in their plane but also with respect to any axis in space. In Fig. 167 let P and Q be the two forces concurrent at A, and R their resultant. Let OX, OY, and OZ be any set of rectangular axes. At point A,

forces P and Q, and their resultant R, may each be resolved into X, Y, and Z components. Since R is the resultant of P and Q, it follows that $R_x = P_x + Q_x$, $R_y = P_y + Q_y$, and $R_z = P_z + Q_z$. It is evident, then, that the moment of R with respect to the X axis is

$$M_x = R_y \times z + R_z \times y$$

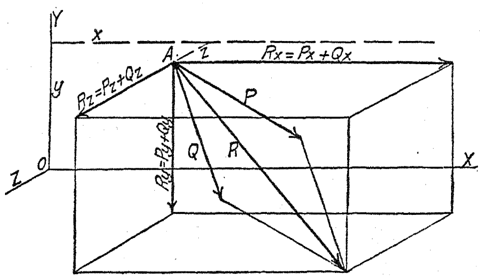


FIG. 167.

From the statement just above, it follows that

$$M_x = (P_y + Q_y)z + (P_z + Q_z)y$$

Since the components P_x and Q_x have no moment with respect to the X axis, the right-hand member of the equation above is the expression for the algebraic sum of the moments of P and Q with respect to the X axis.

In the same manner, it may be shown that

$$M_y = R_x \times z - R_z \times x = (P_x + Q_x)z - (P_z + Q_z)x$$

and

$$M_z = - R_x \times y - R_y \times x = - (P_x + Q_x)y - (P_y + Q_y)x$$

The same relation manifestly holds true for the resultant of R and a third force, and so on for any number of forces.

Therefore: The algebraic sum of the moments of any number of concurrent forces in space with respect to any axis in space is equal to the moment of their resultant with respect to the same axis.

50. Equilibrium of Concurrent Forces in Space: Graphic Solution.—If for any system of noncoplanar, concurrent forces the force polygon (in space) closes, the resultant $R = 0$, and the force system is in equilibrium.

Conversely: If any system of concurrent forces in space is in equilibrium, the force polygon closes.

Since there are only three independent conditions of equilibrium, $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma F_z = 0$, no more than three unknown elements can be determined. These three unknown elements are usually the amounts of three of the forces, the directions being known.

If the force polygon in space closes, the projection of the force polygon on each of the three reference planes closes.

In the solution of problems in equilibrium, the projection of the given system upon some reference plane is drawn, and from the fact that the projection of the force polygon must close, the unknown forces are determined.

Since in any projection there are only two conditions of equilibrium, and therefore no more than two unknown forces can be determined, it is necessary to choose the first plane of projection in such a way that two of the unknown forces coincide in the projection. If the third unknown force is parallel to the plane of projection, its value may be determined directly.

EXAMPLE 1

A shear-legs crane has dimensions and load as shown in Fig. 168(*a*). Determine the stress in *AE* and the stress in *AB*.

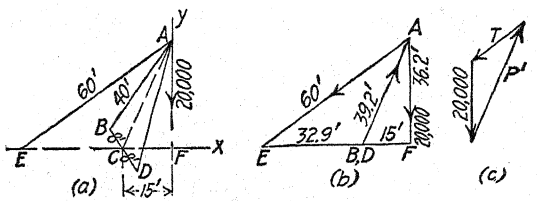


FIG. 168.

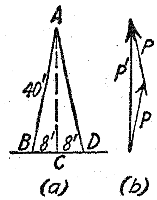


FIG. 169.

Solution.—Take as the plane of projection the vertical plane *AEF*. The force system projected upon this plane is shown in Fig. 168(*b*), which may also be called the free-body diagram of point *A*. In this projection the forces *BA* and *DA* are superimposed. In Fig. 168(*c*) the graphical solution of this projected system is made. The vector *T* gives the stress in *AE* and scales 15,120 lb. The vector *P′* gives the sum of the projected values of the stresses in *BA* and *DA*, 31,500 lb.

In order to determine the stresses in *BA* and *DA*, a view in the plane *ABD* is taken, Fig. 169(*a*). Vector *P′* acts along *CA* and is really the resultant of the stresses in *BA* and *DA*. In order to determine these stresses, vector *P′* is laid down parallel to *CA*, as in Fig. 169(*b*). Through its initial point a line is drawn parallel to *AB*, and through its final point a

line is drawn parallel to AD. Their intersection determines the length of the vectors P, P, which represent the stresses in BA and DA. The scaled value of each is 16,100 lb.

EXAMPLE 2

Figure 170(a) represents a mast 40 ft. high, braced by three cables, each 50 ft. long. The horizontal pull of 2000 lb. may be rotated through 360°. Determine the maximum tension in cable AB as the horizontal pull rotates. Determine the corresponding stress in the mast.

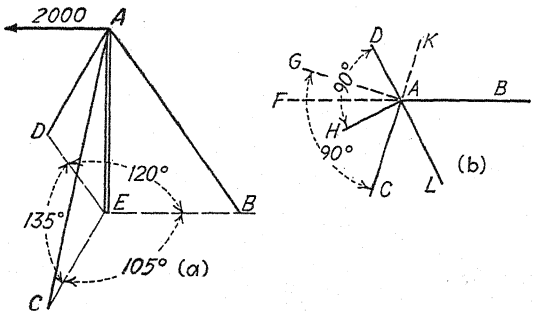


Fig. 170.

Solution.—Figure 170(b) shows the horizontal projection of the forces at A. For any position of the pull between AK and AF, cables AB and AC are acting, and cable AD is idle. As shown in Prob. 6, Art. 20, the stress in AB is a maximum when the line of action of the force is in a plane normal to the other cable, in the position AG. The solution for the horizontal component of AB is shown in Fig. 171(a).

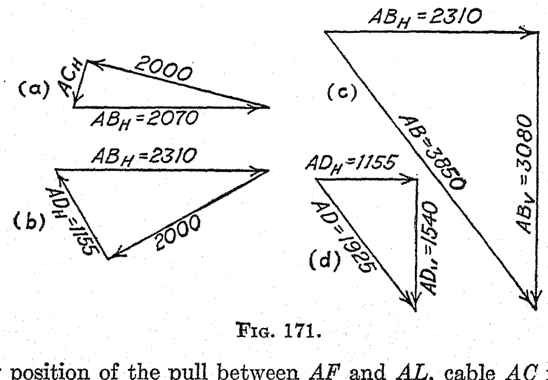


Fig. 171.

For any position of the pull between AF and AL, cable AC is idle, and cable AD comes into action. Another maximum value of the stress in AB is obtained when the line of action of the force is normal to the plane of AD, in the position AH. The solution for the horizontal component of AB is shown in Fig. 171(b). It is easily seen that this maximum value is larger

than the former, since the angle FAH with BA produced is larger than angle FAG.

Of the two positions for a maximum stress in any cable, that one gives the larger value which makes the larger angle with the plane of the cable produced.

Figure 171(*c*) shows the solution for the stress in AB. Vector AB scales 3850 lb. The value of the vertical component AB_v is 3080 lb. Figure 171(*d*) shows the solution for the stress in AD. The vertical component AD_v scales 1540 lb. The compression in the mast must balance these two vertical components. Compression $EA = 3080 + 1540 = 4620$ lb.

Problems

1. A weight of 50 lb. is hung from a horizontal ring 6 ft. in diameter by means of three cords, each 4 ft. long. On the ring the cords are placed 120° apart. Solve for the tension in each cord. *Ans.* 25.2 lb.

2. Solve Prob. 1 if two of the cords are 90° apart and the point of attachment of the third bisects the remaining arc. *Ans.* 22.2 lb.; 22.2 lb.; 31.3 lb.

3. In Fig. 168(*a*), change the length of the leg AE to 70 ft., and distance CF to 18 ft. Solve for the stresses in the three legs.
Ans. $AE = 17{,}000$ lb.; $AB = AD = 16{,}400$ lb.

4. In Fig. 168(*a*), let the 20,000-lb. pull intersect the ground at a point 10 ft. to the right of point F and 5 ft. forward from the line EX. Solve for the stresses in the three legs.
Ans. $AE = 24{,}100$ lb. T; $AB = 12{,}000$ lb. C; $AD = 25{,}200$ lb. C.

5. In Example 2 above, solve for the maximum stress in AC and for the corresponding compression in the mast. *Ans.* 4710 lb.; 6440 lb.

51. Equilibrium of Concurrent Forces in Space: Algebraic Solution.—If for any system of concurrent forces in space the resultant $R = 0$, it must necessarily follow that $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma F_z = 0$ and the force system is in equilibrium.

Conversely: If a system of concurrent forces in space is in equilibrium, the summation of forces along any line equals zero.

This principle gives three independent conditions of equilibrium, from which unknown elements not to exceed three may be determined. These three unknown elements are usually the magnitudes of three of the forces, the directions being known.

If a system of noncoplanar, concurrent forces in equilibrium is projected upon any reference plane, this projection becomes a coplanar system in equilibrium. Since this coplanar system has only two independent conditions of equilibrium, it is necessary to choose the first plane of projection so that two of the unknown forces coincide in the projection.

If a system of concurrent forces in space is in equilibrium, it is also true that the algebraic sum of the moments of all the

forces with respect to any axis in space is equal to zero. If a line intersecting two of the unknown forces is chosen as the first axis of moments, the resulting equation has only one unknown quantity, which is thus determined. If it is desired to write another moment equation, a line intersecting one of the remaining unknown forces should be chosen as the next axis of moments, and the resulting equation will determine the second unknown force. Since only one unknown force remains to be determined, any convenient axis of moments may be chosen for the third equation, or an equation of the summation of forces may be used. Such a combination of the two methods is often the most advantageous.

EXAMPLE 1

Figure 172 represents a haystacking outfit. With a load of 1000 lb. at the middle and a sag of 4 ft. below the horizontal, what are the stresses T_1, T_2, and P?

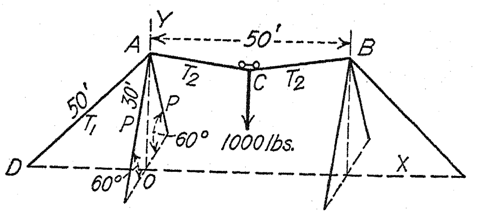


FIG. 172.

Solution.—Consider first the cable ACB and the load as the free body. The stresses T_2 and the weight of the load constitute a coplanar, concurrent system of forces in equilibrium. With a sag of 4 ft. at C, the length $AC = 25.32$ ft. Equation $\Sigma F_y = 0$ gives

$$2T_2 \frac{4}{25.32} = 1000$$

$$T_2 = 3165 \text{ lb.}$$

$$\text{Length } AO = 30 \sin 60° = 25.98 \text{ ft.}$$

$$\text{Length } OD = \sqrt{50^2 - 25.98^2} = 42.7 \text{ ft.}$$

The four forces at A constitute a noncoplanar concurrent system in equilibrium. Equation $\Sigma F_x = 0$ gives

$$\left(3165 \times \frac{25}{25.32}\right) - \frac{42.7}{50} T_1 = 0$$

$$T_1 = 3660 \text{ lb.}$$

By symmetry, the stresses in the two legs are equal. Equation $\Sigma F_y = 0$ gives

$$\left(3660 \times \frac{25.98}{50}\right) + \left(3165 \times \frac{4}{25.32}\right) - \left(2P \times 0.866\right) = 0$$

$$P = 1388 \text{ lb.}$$

If the force system at point A is projected first upon the vertical plane DAO and then upon the vertical plane through the legs, the equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$ give the same equations as those just obtained.

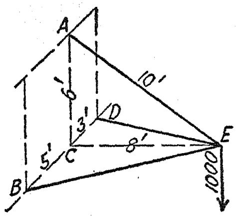


FIG. 173.

EXAMPLE 2

Determine the stresses in the members AE, DE, and BE of the frame shown in Fig. 173.

Solution.—The system of forces at E is a concurrent system in space. If line DB is used as the first axis of moments, equation $\Sigma M = 0$ gives

$$(1000 \times 8) - (AE \times 4.8) = 0$$
$$AE = 1667 \text{ lb.}$$

If next the vertical line through point B is used as the axis of moments, equation $\Sigma M = 0$ gives

$$\left(1667 \times 0.8 \times 5\right) - \left(DE \times 8 \times \frac{8}{8.54}\right) = 0$$
$$DE = 890 \text{ lb.}$$

Equation $\Sigma F_z = 0$ gives

$$\frac{5}{9.42}BE - \left(\frac{3}{8.54} \times 890\right) = 0$$
$$BE = 590 \text{ lb.}$$

As an independent check, the moment equation with respect to the vertical line through D may be written:

$$\left(1667 \times 0.8 \times 3\right) - \left(BE \times 8 \times \frac{8}{9.42}\right) = 0$$
$$BE = 590 \text{ lb.}$$

The trigonometric solution is also easily used with such problems.

Problems

1. A tripod with legs 5 ft. long is set up on a level floor with each leg at the vertex of an equilateral triangle with sides 3 ft. long. Solve for the stresses in the legs caused by a load of 50 lb. on top of the tripod.

Ans. 17.8 lb.

2. Solve Prob. 1 if one of the legs is moved outward a distance of 1.5 ft.

Ans. 24.4 lb.; 16.1 lb.; 16.1 lb.

3. In the haystacking outfit shown in Fig. 172, let the load be moved to a point 15 ft. horizontally from A, and let the distance below the horizontal through A be 3.6 ft. Solve for the stresses AC, BC, AD, and P.

Ans. AC = 3000 lb.; BC = 2930 lb.; AD = 3420 lb.; P = 1430 lb.

4. In the frame shown in Fig. 173, let CD be changed to 7 ft., and let the 1000-lb. force be acting at an angle of 105° with CE in the vertical plane. Solve for the three stresses.

Ans. AE = 1610 lb. T; BE = 707 lb. C; DE = 570 lb. C.

GENERAL PROBLEMS ON CONCURRENT FORCES IN SPACE

1. A gin pole 30 ft. high has three guy cables A, B, and C, which extend from the top of the pole down to the ground at distances of 40, 60, and 75 ft., respectively, from the base of the pole and spaced 120° apart. If the tension in cable A is 8000 lb., solve for the tensions in cables B and C and for the compression in the gin pole. *Ans.* 7160 lb.; 6890 lb.; 10,560 lb.

2. Solve Prob. 1 if the guy cable C is moved around toward A through an angle of 15° and the tension in cable A remains the same.

Ans. 9800 lb.; 8440 lb.; 12,840 lb.

3. Three uniform spheres, each weighing 45 lb., just fit into an equilateral triangular box with smooth vertical sides. A fourth sphere of the same size, but weighing 60 lb., rests on top of the three. What pressure does the box exert on the spheres at each point of contact?

Ans. 65 lb. at bottom; 14.15 lb. at sides.

4. A weight of 800 lb. is suspended by two cables AB and AC, each 25 ft. long. Points B and C are on the same level 16 ft. apart. A horizontal force AD acting normal to the vertical plane through BC holds the weight 10 ft. from the vertical plane through BC. Solve for the stresses AD, AB and AC. *Ans.* $AD = 372$ lb.; $AB = AC = 465$ lb.

5. Solve Prob. 4 if the pull AD acts at an angle of 15° above the horizontal and the distance from the vertical plane through BC is 15 ft.

Ans. $AD = 555$ lb.; $AB = AC = 447$ lb.

6. Solve for all three stresses in the frame shown in Fig. 174.

Ans. $DE = 2400$ lb. C; $AD = 1020$ lb. T; $CD = 730$ lb. T.

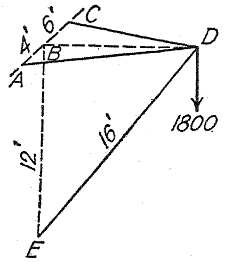


FIG. 174.

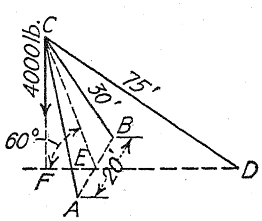


FIG. 175.

7. In Fig. 174, let the 1800-lb. force be acting backward in the plane parallel to the wall ACE, 15° below the horizontal. Solve for the stresses in the three members.

Ans. $DE = 621$ lb. C; $AD = 2230$ lb. T; $CD = 1930$ lb. C.

8. Solve for the stresses in all three members of the shear-legs crane shown in Fig. 175 due to the 4000-lb. load.

Ans. $CD = 3050$ lb. T; $CA = CB = 3060$ lb. C

9. Solve Prob. 8 if the 4000-lb. pull acts forward parallel to line BA.

Ans. $CD = 0$; $CA = 6000$ lb. C; $CB = 6000$ lb. T.

10. Solve Prob. 8 if the 4000-lb. pull acts to the left parallel to DF.

Ans. $CD = 5280$ lb. T; $CA = CB = 1060$ lb. C.

11. A tripod with legs AB, AC, and AD, each 30 ft. long, is set up on level ground with distance $BC = 28$ ft., $CD = 32$ ft., and $DB = 36$ ft. Solve for the stress in each leg caused by a load of 1000 lb. at A.

Ans. $AB = 450$ lb.; $AC = 300$ lb.; $AD = 550$ lb.

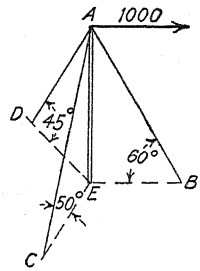


FIG. 176.

12. Figure 176 represents a mast held by three equally spaced guy wires. The pull of 1000 lb., acting horizontally, may rotate about point A. Assuming that the guy wires can take only tensile stress, solve for the stresses in the guy wires and in the mast caused by the 1000-lb. pull when it acts in the plane AEB.

Ans. $AB = 0$; $AC = 1560$ lb. T; $AD = 1410$ lb. T; $AE = 2190$ lb. C.

13. In Fig. 176, solve for the maximum stress in AD and for the corresponding stress in the mast as the 1000-lb. pull rotates about point A in the horizontal plane.

Ans. $AD = 1630$ lb. T; $AE = 1840$ lb. C or 2150 lb. C.

14. Solve Prob. 13 if the 1000-lb. pull is acting at an angle of 20° below the horizontal plane through A.

Ans. $AD = 1540$ lb. T; $AE = 2070$ lb. C or 2370 lb. C.

CHAPTER VI

PARALLEL FORCES IN SPACE

52. Resultant of Parallel Forces in Space, Graphically.—The amount and direction of the resultant of any number of parallel forces in space are given by the algebraic sum of the forces. In order to locate the resultant graphically, the method that is

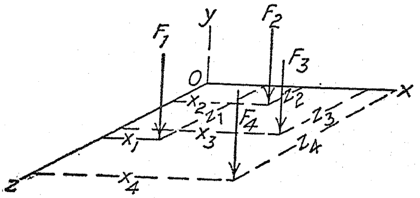


FIG. 177.

usually the simplest is to project the force system upon two reference planes, parallel to the forces and at right angles to each other. The solution for the position of the resultant in these two projections locates it in space.

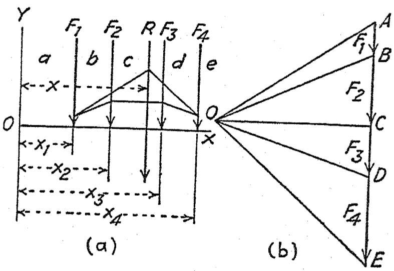


FIG. 178.

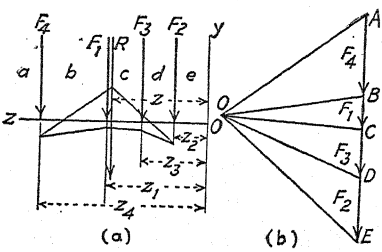


FIG. 179.

Figure 177 represents four forces acting vertically downward parallel to the Y axis. The resultant is of course $R = F_1 + F_2 + F_3 + F_4$. Figure 178 shows the projection of the force system upon the XY plane. By the method shown in Art. 27, the resultant R is located at a distance x from the Y axis. Similarly,

in Fig. 179, the resultant R is located at a distance z from the Y axis. In space, therefore, the resultant R is located a distance x from the YZ plane and a distance z from the XY plane.

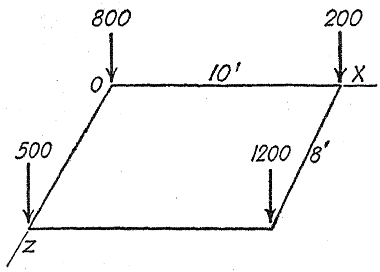


FIG. 180.

Problems

1. Forces are acting vertically downward at the four corners of a rectangular plate as shown in Fig. 180. Combine the four forces graphically.

Ans. $R = 2700$ lb.; $x = 5.19$ ft.; $z = 5.04$ ft.

2. Solve Prob. 1 if the 800-lb. force is reversed in direction.

Ans. $R = 1100$ lb.; $x = 12.73$ ft.; $z = 12.36$ ft.

53. Resultant of Parallel Forces in Space, Algebraically.—In amount and direction the resultant R of any number of parallel forces in space is given by their algebraic sum. The method of locating the *position* of the resultant will now be shown.

In the XY projection, Fig. 178(a), from the principle of Art. 28, the distance x of the resultant R from the Y axis is given by the expression $x = \Sigma Fx/R$. By the principle of projection, this distance x in the XY projection is the same as the distance x of the resultant R from the YZ plane. Similarly, the distance z is given by the expression $z = \Sigma Fz/R$.

Since the same moment equation is obtained for any axis in the YOZ plane parallel to OZ as the one obtained for OZ, it is customary to speak of the *moment of a force with respect to a plane*, meaning thereby the moment of the force with respect to any axis in the given plane normal to the force.

If the moment equation is written for any axis in the YOZ plane at an angle of θ with axis OZ, the same equation is obtained as for the OZ axis with the additional quantity $\cos\theta$ in each term. When this constant quantity is factored out, the moment equation for OZ is obtained, so the general equation of moments for three or more parallel forces in space may be stated:

The algebraic sum of the moments of any number of parallel forces in space with respect to any axis is equal to the moment of their resultant with respect to the same axis.

If the resultant R of a system of parallel forces in space is equal to zero, but the moment of the system with respect to any axis is

not equal to zero, the system is equivalent to a couple, as was discussed in Art. 31.

Problems

1. Solve for the amount, direction, and position of the resultant of the force system shown in Fig. 181.

Ans. $R = 2700$ lb. downward; $x = 0.59$ ft.; $z = 4.19$ ft.

2. Solve Prob. 1 if the 2000-lb. force is reversed in direction.

Ans. $R = 1300$ lb. upward; $x = -7.38$ ft.; $z = 3.62$ ft.

54. Composition of Couples in Space.—If two couples to be combined are in intersecting planes, they may be reduced to couples whose forces are equal each to each. If the couples are then transferred, each in its own plane, so that one force of each lies in the intersection of the two planes in opposite directions, as in Fig. 182, these two forces neutralize each other and may be removed from the system. This leaves the couple with forces FF and arm f in the plane $ABCD$.

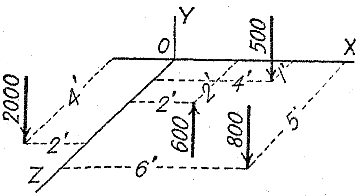


FIG. 181.

If ϕ is the angle of the two planes, and f_1, f_2 the arms of the original couples after being transposed, the arm f is given by

$$f^2 = f_1^2 + f_2^2 - 2f_1f_2 \cos \phi$$

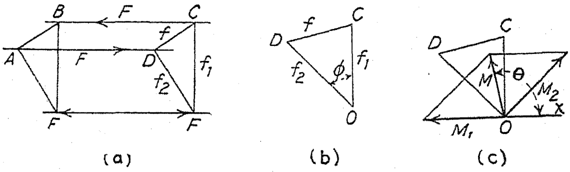


FIG. 182.

These couples may also be combined by means of their vectors. In any plane normal to the line of intersection of the two planes, vector M_1 is laid off normal to the plane of the couple whose arm is f_1 and proportional to the moment of the couple, as shown in Fig. 182(*c*). Vector M_2 is laid off similarly for the other couple. Vector M is the resultant of these two and therefore represents the resultant couple. The amount of the moment is represented to scale by the length of the vector M; the plane of the couple is normal to the vector M, and since the vector points upward, the rotation of the couple is counterclockwise viewed from above.

If the two couples to be combined are in parallel planes, either couple may be transferred to the plane of the other, and the two combined there as explained in Art. 31.

Problems

1. In Fig. 182, let $\phi = 45°$, $M_1 = 800$ lb.-ft., and $M_2 = 1000$ lb.-ft. Determine the resultant couple. *Ans.* $M = 713$ lb.-ft.; $\theta = 97°30'$.

2. Solve Prob. 1 if couple M_2 is reversed in direction.
Ans. $M = 1665$ lb.-ft.; $\theta = 205°10'$.

55. Equilibrium of Parallel Forces in Space: Graphic Solution. If a system of parallel forces in space is in equilibrium, the projection of the system upon any plane constitutes a coplanar system of parallel forces in equilibrium.

Since for any such projected system there are only two conditions of equilibrium, there must not be more than two unknown forces. If there are three unknown forces in the system, the plane of projection must be chosen so that the projections of two of the unknown forces coincide. The third one of the two unknown forces is determined in this projection. The other two must be determined in another projection.

EXAMPLE

A circular table 5 ft. in diameter with three legs equally spaced around the circumference has a load of 100 lb. at the edge and another of 500 lb.

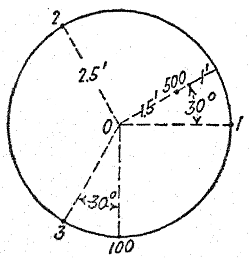


FIG. 183.

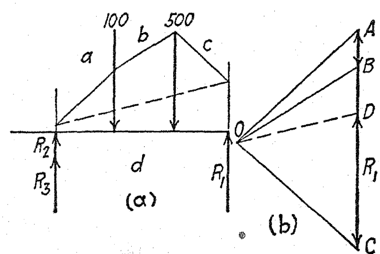


FIG. 184.

1 ft. from the edge as shown in Fig. 183. Determine the three reactions R_1, R_2, and R_3.

Solution.—There are three unknown forces, but the projection of the forces on a vertical plane parallel to O-1 constitutes a coplanar system in equilibrium and so can be solved. Figure 184 shows the solution for R_1 which scales 374 lb. Vector DA represents $R_2 + R_3$ and scales 226 lb.

Figure 185(a) is the projection of the force system upon a vertical plane through R_2R_3. Since R_1 is already determined, there are only two unknown

forces, and these are determined in Fig. 185. DE scales 84 lb., and EA scales 142 lb.

Problems

1. A triangular steel plate ABC, ½ in. thick, has side AB 8 ft. long, side BC 6 ft. long, and side CA 10 ft. long. It is supported in a horizontal position by reactions at the corners. Solve for the reactions due to its own weight (490 lb./cu. ft.) and to a load of 500 lb. placed 1 ft. from side AB and 3 ft. from side BC. (The center of gravity of a triangular plate is at a point one-third the altitude from any base.)

Ans. $A = 351$ lb.; $B = 392$ lb.; $C = 247$ lb.

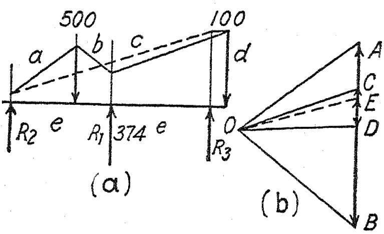


FIG. 185.

2. Locate the position of the 500-lb. load on the steel plate described in Prob. 1 so that reaction A is one-half the total weight, and reactions B and C are each one-fourth the total weight.

Ans. 1.01 ft. from AB; 5.31 ft. from BC.

56. Equilibrium of Parallel Forces in Space: Algebraic Solution.—If for any system of parallel forces in space the resultant R is equal to zero and the moment M with respect to any axis is equal to zero, the system is in equilibrium.

Conversely: If a system of parallel forces in space is in equilibrium, the resultant $R = 0$ and $\Sigma M = 0$ with respect to any axis in space.

Since there are only three independent equations, one of summation of forces and two of moments, no more than three unknown elements can be determined. These are usually the amounts of three of the forces, their lines of action being known.

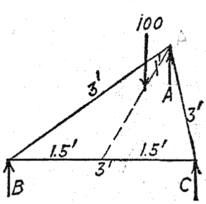


FIG. 186.

EXAMPLE

A horizontal equilateral triangular plate ABC, 3 ft. on a side, is supported at the vertices. What are the three reactions due to a load of 100 lb. acting at a point on the median line 1 ft. from vertex A, as shown in Fig. 186?

Solution.—The altitude of the triangle is 2.6 ft. The distance from the base BC to the load is 1.6 ft. Equation $\Sigma M = 0$ for axis through the edge BC gives

$$100 \times 1.6 = 2.6\ A$$
$$A = 61.5 \text{ lb.}$$

By symmetry, reaction B = reaction C. Equation $\Sigma F = 0$ gives

$$\begin{aligned} B + C + 61.5 &= 100 \\ B + C &= 38.5 \\ B = C &= 19.25 \text{ lb.} \end{aligned}$$

Problems

1. A rectangular table 12 ft. long and 5 ft. wide is supported by legs A and B at the corners at one end and by leg C at the other, 1 ft. from the middle toward side B. If the table weighs 300 lb., and its center of gravity is at the middle, what are the three reactions?

Ans. A = 105 lb.; B = 45 lb.; C = 150 lb.

2. If a load of 1200 lb. is placed on the table described in Prob. 1, at a point 3 ft. from end AB and 1 ft. from side B what are the total reactions due both to the load and to the weight of the table?

Ans. A = 255 lb.; B = 795 lb.; C = 450 lb.

GENERAL PROBLEMS ON PARALLEL FORCES IN SPACE

1. Four weights on a horizontal rectangular plate 5 ft. square have the following amounts and coordinates: 60 lb. at (3 ft., 4 ft.); 40 lb. at (1 ft., 5 ft.); 80 lb. at (4 ft., 1 ft.); 50 lb. at (1 ft., 1 ft.). Solve for the amount and position of the resultant. *Ans.* R = 230 lb. at (2.565 ft., 2.478 ft.).

2. If the plate described in Prob. 1 is supported at three points A at (0 ft., 0 ft.), B at (5 ft., 2 ft.), and C at (2 ft., 5 ft.), determine the three reactions due to the weights.

Ans. A = 64.3 lb.; B = 86.2 lb.; C = 79.5 lb.

3. In Prob. 2, how far in along the diagonal must the reaction at A be moved in order to increase it to 100 lb.? *Ans.* 1.768 ft.

4. A horizontal equilateral triangular plate ABC, 6 ft. on a side, has weights of 20, 60, and 120 lb. at the middle points of sides AB, BC, and CA, respectively. Locate the resultant from each side.

Ans. 2.34 ft. from AB; 1.82 ft. from BC; 1.04 ft. from CA.

5. If the plate described in Prob. 4 is supported at the vertices, solve for the reactions due to the weights. *Ans.* A = 70 lb.; B = 40 lb.; C = 90 lb.

6. In Prob. 5, how far in along the median line must the reaction A be moved in order that it may be increased to 100 lb.? *Ans.* 1.56 ft.

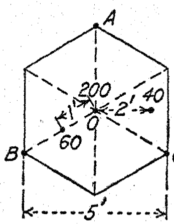


FIG. 187.

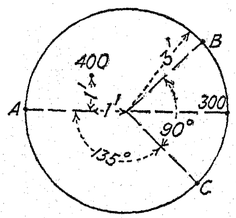


FIG. 188.

7. Solve Prob. 5 if reaction B is moved 1 ft. toward A along the side BA.

Ans. A = 62 lb.; B = 48 lb.; C = 90 lb.

8. A hexagonal table supported at A, B, and C, Fig. 187, carries three loads as shown. Solve for the reactions.

Ans. $A = 93$ lb.; $B = 98$ lb.; $C = 109$ lb.

9. In Fig. 187, solve for the load at the vertex between A and B that will reduce the reaction at C to zero. *Ans.* 327.5 lb.

10. Figure 188 shows the top view of a circular plate weighing 500 lb. that is supported at points A, B, and C and carries two additional loads as shown. Compute the three reactions.

Ans. $A = 400$ lb.; $B = 494$ lb.; $C = 306$ lb.

11. Solve Prob. 10 if the support at C is moved around toward A through an angle of 15°. *Ans.* $A = 364$ lb.; $B = 545$ lb.; $C = 291$ lb.

CHAPTER VII

NONCONCURRENT, NONPARALLEL FORCES IN SPACE

57. Resultant of Nonconcurrent, Nonparallel Forces in Space, Algebraically.—The simplest resultant system to which any given system of nonconcurrent, nonparallel forces in space may be reduced is a single resultant force R and a single resultant couple M. Any arbitrary point may be selected as the point through which the resultant R is to act, and the amount of the resultant couple M will differ for each different line of action of the resultant R. At any convenient point on its own line of action, each force of the system may be resolved into its X, Y, and Z components by one of the methods of Art. 46. These components may then be replaced by X, Y, and Z forces at the selected point together with couples in the three reference planes by the method of Art. 32. After this operation has been performed for all the forces, the concurrent system at the selected point may be recombined into a single resultant, the value of which is given by the expression

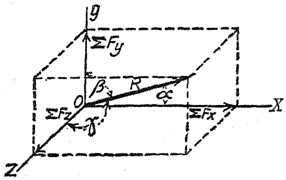


FIG. 189.

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$$

as shown in Fig. 189. The direction cosines are given by the following expressions:

$$\text{os } \alpha = \frac{\Sigma F_x}{R}; \quad \cos \beta = \frac{\Sigma F_y}{R}; \quad \cos \gamma = \frac{\Sigma F_z}{R}$$

The algebraic sum of the moments of the couples referred to above with respect to the X, Y, and Z axes in turn gives M_x, M_y, and M_z, which may be represented by their vectors. The vector of the resultant couple is given by the expression

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}$$

Also

$$\cos \alpha_1 = \frac{M_x}{M}; \quad \cos \beta_1 = \frac{M_y}{M}; \quad \cos \gamma_1 = \frac{M_z}{M}$$

In general, then, a system of this kind tends to produce a translation of the body acted upon in the direction of R and a rotation in the plane of the resultant couple M.

EXAMPLE

Combine the forces shown in Fig. 190 into a force at O and a couple.

Solution.—The 200-lb. force will be resolved at A into its three rectangular components. The x, y, and z dimensions of the rectangular

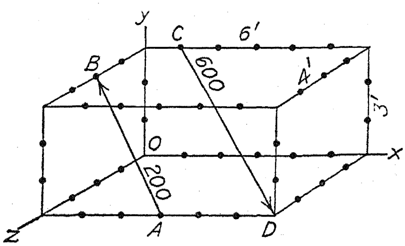


FIG. 190.

parallelepiped of which the line of action of this force is the diagonal are 3 3, and 2 ft., respectively. The length of the diagonal is

$$AB = \sqrt{9 + 9 + 4} = 4.69 \text{ ft.}$$

$$F_x = -200 \times \frac{3}{4.69} = -128 \text{ lb.}$$

$$F_y = 200 \times \frac{3}{4.69} = 128 \text{ lb.}$$

$$F_z = -200 \times \frac{2}{4.69} = -85.3 \text{ lb.}$$

If these components at A are replaced by equal forces at O and couples the couples become

$$M_x = -128 \times 4 = -512 \text{ lb.-ft.}$$
$$M_y = (85.3 \times 3) - (128 \times 4) = -256 \text{ lb.-ft.}$$
$$M_z = 128 \times 3 = 384 \text{ lb.-ft.}$$

The 600-lb. force will be resolved at C into its three rectangular components. The x, y, and z dimensions of the rectangular parallelepiped of which the line of action of this force is the diagonal are 5, 3, and 4 ft., respectively. The length of the diagonal is

$$CD = \sqrt{25 + 9 + 16} = 7.07 \text{ ft.}$$

$$F_x = 600 \times \frac{5}{7.07} = 424.2 \text{ lb.}$$

$$F_y = -600 \times \frac{3}{7.07} = -254.5 \text{ lb.}$$

$$F_z = 600 \times \frac{4}{7.07} = 339.4 \text{ lb.}$$

When these forces at C are replaced by equal forces at O and couples, the couples become

$$M_x = 339.4 \times 3 = 1018.2 \text{ lb.-ft.}$$
$$M_y = -339.4 \times 1 = -339.4 \text{ lb.-ft.}$$
$$M_z = -424.2 \times 3 - 254.5 \times 1 = -1527.1 \text{ lb.-ft.}$$

The forces may now be recombined.

$$\Sigma F_x = -128 + 424.2 = 296.2 \text{ lb.}$$
$$\Sigma F_y = 128 - 254.5 = -126.5 \text{ lb.}$$
$$\Sigma F_z = -85.3 + 339.4 = 254.1 \text{ lb.}$$
$$R = \sqrt{168{,}305} = 410.3 \text{ lb.}$$
$$\alpha = \cos^{-1} \frac{296.2}{410.3} = 43°50'$$
$$\beta = \cos^{-1} -\frac{126.5}{410.3} = 107°55'$$
$$\gamma = \cos^{-1} \frac{254.1}{410.3} = 51°40'$$

In a similar manner, the couples may be combined into their resultant couple M.

$$\Sigma M_x = -512 + 1018.2 = 506.2 \text{ lb.-ft.}$$
$$\Sigma M_y = -256 - 339.4 = -595.4 \text{ lb.-ft.}$$
$$\Sigma M_z = 384 - 1527.1 = -1143.1 \text{ lb.-ft.}$$
$$M = \sqrt{1{,}919{,}740} = 1386 \text{ lb.-ft.}$$
$$\alpha_1 = \cos^{-1} \frac{506.2}{1386} = 68°30'$$
$$\beta_1 = \cos^{-1} -\frac{595.4}{1386} = 115°30'$$
$$\gamma_1 = \cos^{-1} \frac{1143.1}{1386} = 145°40'$$

Problems

1. Combine the forces shown in Fig. 191 into a force at O and a couple. The sides of the cube are 2 ft. long.

Ans. $R = 244$ lb.; $\alpha = 76°35'$; $\beta = 164°0'$; $\gamma = 81°33'$; $M = 527$ lb.-ft.; $\alpha_1 = 68°10'$; $\beta_1 = 90°$; $\gamma_1 = 158°20'$.

2. Solve Prob. 1 using point A as the point through which the resultant R is to act.

Ans. R is unchanged; $M = 360$ lb.-ft.; $\alpha_1 = 163°50'$; $\beta_1 = 96°40'$; $\gamma_1 = 75°10'$.

58. Resultant of Nonconcurrent, Nonparallel Forces in Space, Graphically.—By the method of Art. 46, each force vector of a system of nonconcurrent, nonparallel forces in space may be resolved at any point into its X, Y, and Z components. From all these, ΣF_x, ΣF_y, and ΣF_z may be obtained, and the resultant R is given by the diagonal of the rectangular parallelepiped formed upon these three vectors.

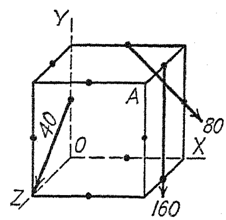


FIG. 191.

Any convenient point may be chosen as the point through which the resultant is to act. Through this point as an origin, the X, Y, and Z axes are drawn. From the scaled values of the component vectors and their lever arms with respect to these selected axes, the values of M_x, M_y, and M_z are obtained and represented graphically by couple vectors. The diagonal of the rectangular parallelepiped constructed upon these moment vectors gives the resultant moment of the system.

The resultant of such a system of forces, then, consists of a resultant force R acting through the selected origin, and a resultant couple M.

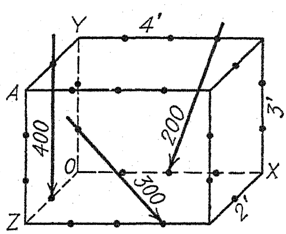


FIG. 192.

Problems

1. Combine the forces shown in Fig. 192 into a force at O and a couple.

Ans. $R = 711$ lb.; $\alpha = 75°40'$; $\beta = 160°20'$; $\gamma = 77°0'$; $M = 838$ lb.-ft.; $\alpha_1 = 48°0'$; $\beta_1 = 90°$; $\gamma_1 = 137°45'$.

2. Solve Prob. 1 using point A as the point through which the resultant R is to act.

Ans. R is unchanged; $M = 1312$ lb.-ft.; $\alpha_1 = 163°50'$; $\beta_1 = 105°40'$; $\gamma_1 = 93°55'$.

59. Equilibrium of Nonconcurrent, Nonparallel Forces: Algebraic Solution.—If for a system of nonconcurrent, nonparallel forces in space the resultant force R through any point is zero, and the corresponding resultant couple M is also zero, the effect of the system is zero, and it is therefore in equilibrium.

Conversely: If a system of nonconcurrent, nonparallel forces in space is in equilibrium, the resultant force R is zero, and the resultant moment M is zero.

If $R = 0$, it follows that

$$\Sigma F_x = 0, \ \Sigma F_y = 0, \text{ and } \Sigma F_z = 0$$

If $M = 0$, it follows also that

$$\Sigma M_x = 0, \ \Sigma M_y = 0, \text{ and } \Sigma M_z = 0$$

These conditions of equilibrium may be taken in sets of three, as follows:

$$\begin{aligned} \Sigma F_x &= 0, \ \Sigma F_y = 0, \ \Sigma M_z = 0 \\ \Sigma F_x &= 0, \ \Sigma F_z = 0, \ \Sigma M_y = 0 \\ \Sigma F_y &= 0, \ \Sigma F_z = 0, \ \Sigma M_x = 0 \end{aligned}$$

It will be seen that the first set gives the conditions necessary for equilibrium of a coplanar system of forces in the XY plane; the second set gives the conditions necessary for equilibrium of a coplanar system of forces in the XZ plane; the third set gives the conditions necessary for equilibrium of a coplanar system of forces in the YZ plane. Also, ΣF_x and ΣF_y are the X and Y components of the projections of the forces on the XY plane, and ΣM_z is the moment of these forces in that plane.

If a system of nonconcurrent, nonparallel forces in space is in equilibrium, the projection of these forces on any plane constitutes a system of forces in equilibrium.

By means of these principles, unknown forces not to exceed the number of equations may be determined in any system that is known to be in equilibrium.

EXAMPLE

Figure 193 shows three views of a simple windlass. It is required to determine P, A, and B for the position shown, A and B being the reactions at the supports.

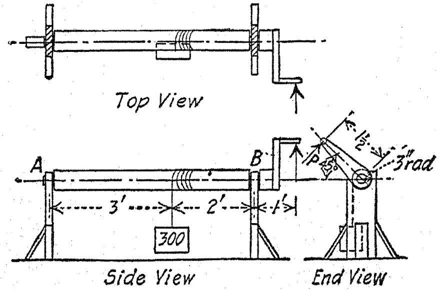


FIG. 193.

Solution.—The free-body diagram for the top view, the projection of the figure on the XZ plane, is shown in Fig. 194(a). That for the side view, the projection on the XY plane, is shown in Fig. 194(b). That for the end view, the projection on the YZ plane, is shown in Fig. 194(c). In these diagrams A is replaced by its horizontal and vertical components A_z and A_y. Also B

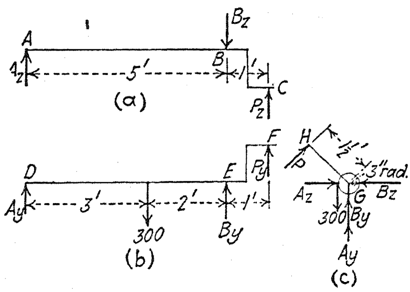


FIG. 194.

is replaced by its horizontal and vertical components B_z and B_y. The equation $\Sigma M_G = 0$ for Fig. 194(c) gives

$$P \times 18 = 300 \times 3$$
$$P = 50 \text{ lb.}$$

The four other unknown forces in this projection cannot be determined. With P known, the unknown forces in Fig. 194(a) can now be determined.

$$P_z = P \cos 45° = 35.35 \text{ lb.}$$

Equation $\Sigma M_A = 0$ gives

$$35.35 \times 6 = B_z \times 5$$
$$B_z = 42.42 \text{ lb.}$$

Equation $\Sigma M_B = 0$ gives

$$35.35 \times 1 = A_z \times 5$$
$$A_z = 7.07 \text{ lb.}$$

In Fig. 194(b), $P_y = P \sin 45° = 35.35$ lb.
Equation $\Sigma M_D = 0$ gives

$$(35.35 \times 6) + (B_y \times 5) = 300 \times 3$$
$$B_y = 137.6 \text{ lb.}$$

Equation $\Sigma M_E = 0$ gives

$$(35.35 \times 1) + (300 \times 2) = A_y \times 5$$
$$A_y = 127.1 \text{ lb.}$$

The reaction $A = \sqrt{A_z^2 + A_y^2}$
$A = \sqrt{7.07^2 + 127.1^2}$
$A = 127.5$ lb.

The angle θ_A with the vertical is given by

$$\theta_A = \tan^{-1}\frac{7.07}{127.1} = \tan^{-1} 0.0555 = 3°10'$$

Similarly,

$$B = \sqrt{42.42^2 + 137.6^2}$$
$$B = 144 \text{ lb.}$$
$$\theta_B = \tan^{-1}\frac{42.42}{137.6} = \tan^{-1} 0.308 = 17°06'$$

If force P had an X component, it would be balanced in the X direction by reactions A_x and B_x. This is a redundant system, since either reaction would be sufficient, so some assumption would have to be made as to the distribution of the reactions. For instance, the assumption may be made that A_x and B_x are equal; that $B_x = 2A_x$; or that $A_x = 0$ and $B_x = P_x$. It should also be noted that an X component of force P would change the values of A_z, A_y, B_z, and B_y.

Problems

1. Solve for the horizontal and vertical components of the reactions on the windlass of the example above when the handle has been turned forward through an angle of 75° from the position shown in Fig. 193.

Ans. $A_y = 115$ lb.; $B_y = 210$ lb.; $A_z = 8.7$ lb.; $B_z = 52$ lb.

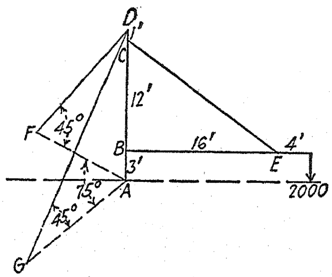


FIG. 195.

2. The boom and the mast of the crane shown in Fig. 195 weigh 50 lb. per linear foot. Neglecting the weights of the other members, solve for the vertical and horizontal components of the reactions at A and B and for the stresses in CE, DF, and DG when the plane of the boom bisects the angle GAF.

Ans. $A_y = 7740$ lb.; $A_x = 3125$ lb.; $B_y = 125$ lb.; $B_x = 4167$ lb.; $CE = 5210$ lb.; $DF = DG = 2780$ lb.

3. If the boom BE of the crane shown in Fig. 195 is rotated about the mast AD, locate its position to cause the maximum tension in member DF. Solve for this maximum tension and for the stress in member DG. It is assumed that the back stays can take compression.

Ans. 4590 lb. T; 1185 lb. C.

60. Equilibrium of Nonconcurrent, Nonparallel Forces in Space: Graphic Solution.—It was shown in Art. 59 that for a system of nonconcurrent, nonparallel forces in space that is in equilibrium, the projection of the force system upon any plane constitutes a coplanar system of forces in equilibrium. By applying the graphic conditions of equilibrium for such a system, the unknown

forces in the system may be determined, provided the unknown elements do not exceed the number of independent conditions of equilibrium.

EXAMPLE

Figure 196(a) shows the dimensions, position, and loading of a derrick. Determine the external reactions due to a load of 1200 lb.

Solution.—The external forces on the derrick constitute a nonconcurrent, nonparallel system in space; but by taking different parts of the derrick as

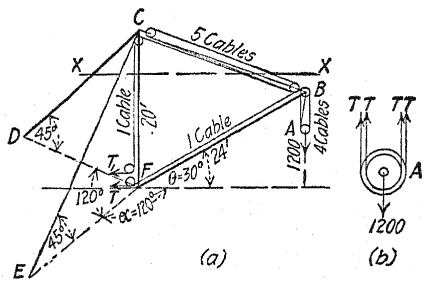


Fig. 196.

free bodies in turn, only concurrent systems need be considered. The first free body to be considered is the pulley at A, shown in its free-body diagram in Fig. 196(b). If friction is neglected, the tension in any cable is constant throughout its length. Then equation $\Sigma F_y = 0$ gives

$$4T = 1200 \text{ (neglecting the slight angularity)}$$
$$T = 300 \text{ lb.}$$

The pin at B has forces acting upon it as shown in Fig. 197(a), forces T_1 and P being unknown. Figure 197(b) shows the graphic solution, from which T_1 scales 268 lb. and P scales 1740 lb.

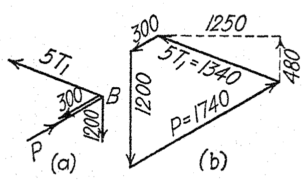


Fig. 197.

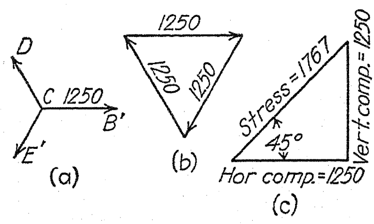


Fig. 198.

Consider next a section made by the horizontal plane XX, Fig. 196(a), and let the part above the plane be taken as the free body. The system of forces acting on this free body is concurrent but not coplanar. The horizontal projections of these forces, however, constitute a coplanar, concurrent system in equilibrium and so may readily be solved. From Fig. 197(b) the horizontal component of $5T_1$ is 1250 lb. From Fig. 198(a) and (b) it is

seen that the horizontal components of the stresses in CD and CE are each 1250 lb. The stress in each, as shown in Fig. 198(c), is 1767 lb.

The compression in the mast is determined by considering the vertical forces on the free body above plane XX, Fig. 196(a). Let the compressive stress in the mast be called V. The vertical component of the stress in CB is 480 lb., as shown in Fig. 197(b). The vertical component of the stress in CD is 1250 lb. The vertical component of the stress in CE is 1250 lb., as shown in Fig. 198(c). Equation $\Sigma F_y = 0$ gives

$$V - 1250 - 1250 - 268 - 480 = 0$$
$$V = 3248 \text{ lb.}$$

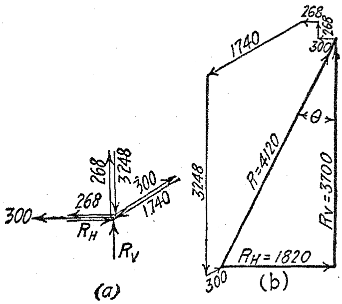


FIG. 199.

Next take the socket at F as the free body. Figure 199(a) shows the free-body diagram, and Fig. 199(b) shows the force diagram, from which R_H scales 1820 lb., R_V scales 3700 lb., and R scales 4120 lb. The angle θ with the vertical scales 26°10′.

If the boom BF is rotated about the mast CF, the stresses in the stiff legs CD and CE will vary. If the boom is rotated toward CE, the stress in CE will decrease, and that in CD will increase. Figure 200(b) shows the free body above plane XX projected on the horizontal plane, the vertical plane

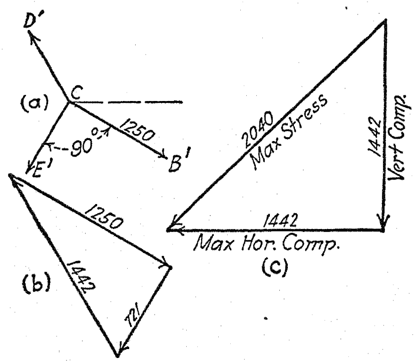


FIG. 200.

through the boom being 90° from the vertical plane through EC. It is seen from Fig. 200(b) that this position of the boom causes a maximum horizontal component in CD, since vector CB' remains constant in amount and CE' and CD' remain fixed in direction. Figure 200(c) shows the solution for the stress in CD.

If the boom is rotated so that CB' is in line with CD', the horizontal component of CD' is only 1250 lb., and the stress in CE becomes zero.

If the boom is rotated nearer to CE, the stress in CE becomes compression. The stress in CD decreases and becomes zero when the boom is in the plane of CE.

If framed members are inserted at DF and EF, they will carry the horizontal components of the stresses in CD and CE to the foot of the mast.

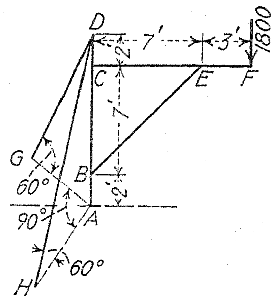


FIG. 201.

Problems

1. In the crane shown in Fig. 201, solve for the horizontal and vertical components of the reactions at A and C and for the stress in BE. Solve also for the stresses in members DG and DH when the plane of the boom bisects the angle GAH.

Ans. A_x = 1640 lb.; A_y = 5800 lb.; C_x = 2570 lb.; C_y = 770 lb.; BE = 3640 lb.; DG = DH = 2315 lb.

2. In Fig. 201, solve for the maximum stress in member DG as the boom rotates about the mast. *Ans.* 3270 lb.

GENERAL PROBLEMS ON NONCONCURRENT, NONPARALLEL FORCES IN SPACE

1. In Fig. 202, reduce the 200- and the 300-lb. forces to a force at O and a couple.

Ans. R = 149 lb.; α = 128°0′; β = 45°10′; γ = 69°30′; M = 1133 lb.-ft.; α_1 = 137°45′; β_1 = 132°05′; γ_1 = 87°20′.

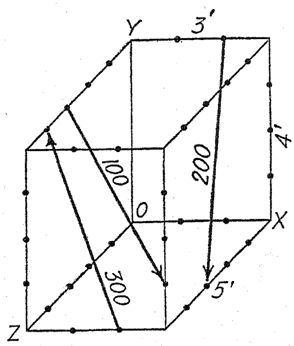


FIG. 202.

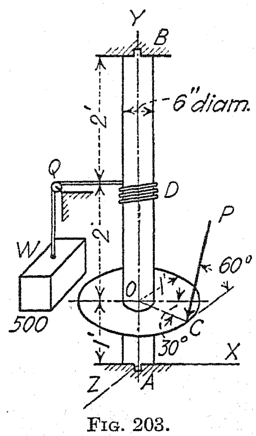


FIG. 203.

2. Reduce all three of the forces shown in Fig. 202 to three rectangular forces at O and three rectangular couples.

Ans. R_x = −28 lb.; R_y = 41 lb.; R_z = 95 lb.; M_x = −478 lb.-ft.; M_y = −567 lb.-ft.; M_z = −203 lb.-ft.

3. Figure 203 represents a vertical windlass, supported in sockets at A and B. A cord is wrapped around the windlass at D, passing horizontally parallel to the X axis to the pulley Q, thence down to the weight W. The pull in the cord QD is balanced by the pressure of the force P at the rim of a pulley 2 ft. in diameter. Radius OC is 30° forward from the XY plane. Force P acts normal to OC at an angle of 60° with the horizontal. Neglecting the weight of the windlass and the friction at the bearings, determine the amount of force P and the rectangular components of the reactions at A and B.

Ans. $P = 250$ lb.; $A_x = 288$ lb.; $A_y = 217$ lb.; $A_z = 65$ lb.; $B_x = 275$ lb.; $B_z = 43$ lb.

4. In Prob. 3, let force P be acting at an angle of 15° with the horizontal, and let radius OC be 45° forward from the XY plane. With all other data the same, solve for force P and the rectangular components of the reactions at A and B.

Ans. $P = 129$ lb.; $A_x = 275$ lb.; $A_y = 33$ lb.; $A_z = 66$ lb.; $B_x = 313$ lb.; $B_z = 22$ lb.

5. If the weight on the windlass shown in Fig. 203 is supported by means of a couple consisting of two forces applied tangentially at the circumference of the pulley, determine the amount of each force and the reactions at A and B. *Ans.* 62.5 lb.; $A_x = 200$ lb.; $B_x = 300$ lb.

6. If the weight on the windlass shown in Fig. 203 is supported by means of two belt pulls T_2 and T_1 acting forward parallel to the Z axis on the pulley at O, and if $T_2 = 1.5T_1$, solve for the components of the reactions at A and B.

Ans. $T_2 = 375$ lb.; $T_1 = 250$ lb.; $A_x = 200$ lb.; $B_x = 300$ lb.; $A_z = 500$ lb.; $B_z = 125$ lb.

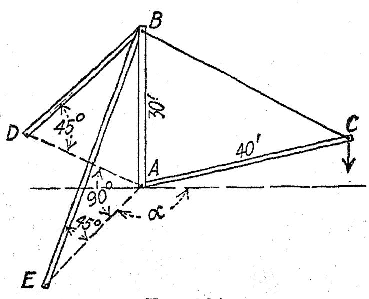


FIG. 204.

7. The boom of the stiff-leg derrick shown in Fig. 204 has a range of position in a vertical plane from the horizontal to within 20° of the vertical. Determine the position of the boom for the maximum stress in BC. For this position of the boom in the vertical plane, determine the value of angle α for the maximum compression in BE. Do the same for the maximum tension. If the load at C is 18,000 lb., determine the maximum stresses in BC, AC, BE, and BA.

Ans. $BC = 30{,}000$ lb.; $AC = 24{,}000$ lb.; $BE = 33{,}940$ lb.; $BA = 51{,}940$ lb.

8. Figure 205 represents a dipper dredge with dimensions as shown. The boom CG weighs 32,000 lb., with its center of gravity at the middle. The handle HF weighs 4000 lb., with its center of gravity at the middle. In the position shown, it is at an angle of 15° with the vertical. The dipper and load weigh 9600 lb., with their center of gravity at point F. BD is an A-frame 40 ft. in altitude and spaced 30 ft. apart at the base. In the filling position, consider the pressure to be 8000 lb. applied at right angles to the

handle. Solve for the stresses in the cable FG, the cables DG and DA, the pin reactions at E and C, and the compression in each leg of the A-frame DB.

Ans. FG = 19,650 lb.; DG = 32,800 lb.; DA = 49,360 lb.; E = 3290 lb. at 62°30′ with X; C = 72,080 lb. at 39°40′ with X; DB = 19,250 lb. in each leg.

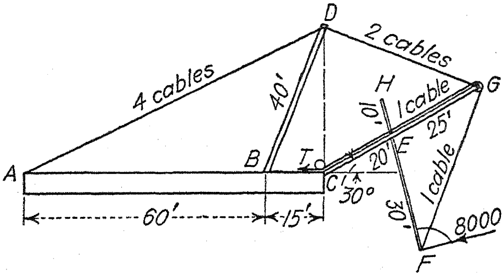


FIG. 205.

9. In the derrick shown in Fig. 206, four guy cables, each at an angle of 30° with the ground, support a mast 65 ft. high. The boom is 95 ft. long. When the boom is in the horizontal position, and a load of 50 tons is being lifted, determine the stresses in the cables FG, cable EG, and boom EG. Determine also the maximum stress that can come upon any one of the four guy cables as the boom is rotated horizontally.

Ans. FG = 29,530 lb. in each cable; cable EG = 20,000 lb.; boom EG = 166,000 lb.; maximum stress in guy cable = 168,600 lb.

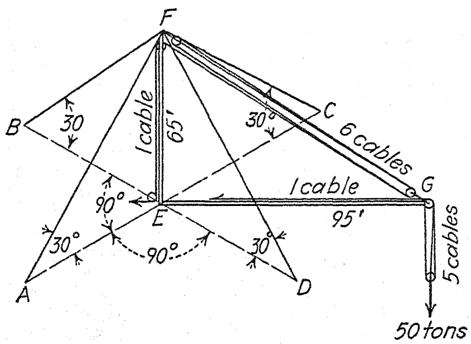


FIG. 206.

10. Solve for the same stresses as those asked for in Prob. 9 if the cables FG are shortened until the boom makes an angle of 60° with the horizontal.

Ans. FG = 12,960 lb. in each cable; cable EG = 20,000 lb.; boom EG = 166,000 lb.; maximum stress in guy cable = 84,400 lb.

CHAPTER VIII

FRICTION

61. Static and Kinetic Friction.—If a block rests upon a horizontal supporting surface, the weight of the block and the resistance of the surface are the two forces acting upon the block. If these distributed forces are considered to be acting at the center of the area of contact, they may be represented by W and N, Fig. 207(a). If a small horizontal force P is applied to the block, and it is still at rest, the force to balance P is the resistance of the supporting plane parallel to P, tangential to the surface, as shown

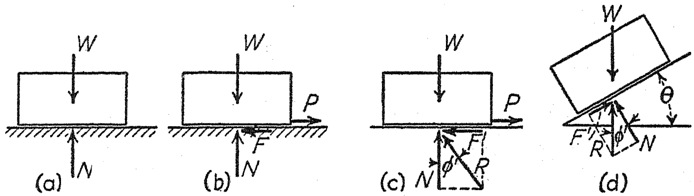


FIG. 207.

in Fig. 207(b). This resistance is called *friction* and is denoted by F.

If the force P is increased gradually, it will reach a certain value that the friction F can no longer balance, and the block will move. While the block is at rest, the friction is called *static friction*. The highest value of the static friction, that when motion is just impending, is called the *limiting friction* and will be denoted by F'. After motion begins, the friction decreases and is called *kinetic friction*, or friction of motion. If the block is moving or tending to move over a supporting surface, the friction of the supporting surface is opposite to the direction of the motion.

Adhesion should not be confused with friction. Adhesion is the attraction between two surfaces in contact. It depends upon the area in contact and is independent of the pressure. Friction is independent of the area and varies as the pressure. For nearly all problems in engineering, adhesion may be neglected.

If the two surfaces in contact are hard and well polished, the frictional resistance becomes very small but never reaches zero. If the friction could reach zero, the surface would be the ideal *smooth* surface, for which the resistance would be normal to the surface of contact. In some problems in engineering, the friction is very small compared with the other forces acting and may be neglected in the solution without appreciable error.

62. Coefficient of Friction and Angle of Friction.—The *coefficient of static friction*, denoted by f, is the ratio of the limiting friction F' to the normal pressure N.

$$f = \frac{F'}{N}$$

The frictional force F and the normal reaction N acting on the block in Fig. 207(c) may be combined into their resultant R. It is evident that the resultant R must always lean from the normal in the direction to oppose motion.

If ϕ is the angle between the resultant reaction and the normal, it is plain from Fig. 207(d) that $F/N = \tan \phi$. The maximum value of ϕ corresponding to F' is denoted by ϕ' and is called the *angle of friction*. It is evident that $f = \tan \phi'$.

If the surface upon which the block rests is inclined at an angle θ with the horizontal, and no force but the pull of gravity and the reaction of the surface acts upon the block, the angle at which slipping is impending is θ', the *angle of repose*. In Fig. 207(d), R is equal and opposite to W and acts at the angle ϕ' with the normal, since slipping impends. From the geometry of the figure, angle θ' = angle ϕ'.

If the value of the angle ϕ' for two given surfaces is known, and slipping is impending, the resultant reaction becomes known in direction.

The coefficient of static friction for two surfaces may be determined experimentally by finding the pull P necessary to start a weight W on a horizontal plane or by finding the angle of inclination of the plane at which motion is impending for the weight resting upon it.

The *coefficient of kinetic friction* is the ratio of the kinetic friction F to the normal pressure N and is also denoted by f.

$$f = \frac{F}{N}$$

The coefficient of kinetic friction may be determined by finding the pull P necessary to keep a weight W moving uniformly on a horizontal plane or by finding the angle of inclination of the plane at which the motion of the weight upon it is uniform.

As would be expected, there are great variations in the values of the coefficients so obtained. The following table gives the range of values for the coefficient of static friction for a few materials.

The corresponding coefficients of kinetic friction are 20 to 40 per cent less than the values for static friction.

Substances	Static f	Substances	Static f
Wood on wood.........	0.30–0.70	Leather on wood......	0.25–0.50
Metal on metal.........	0.15–0.30	Leather on metal......	0.30–0.60
Wood on metal.........	0.20–0.60	Stone on stone........	0.40–0.65

EXAMPLE

A block weighing 500 lb. rests upon two wedges which, in turn, rest upon a horizontal plane surface as shown in Fig. 208(a). If the angle of the

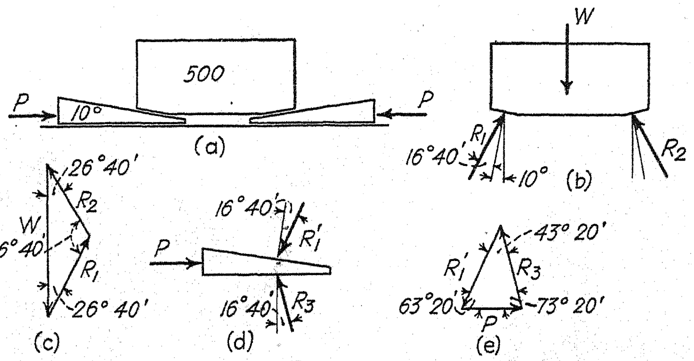


FIG. 208.

wedges is 10°, and the coefficient of friction is 0.30, what are the forces P, P required to force the wedges under the block?

Graphic Solution.—The angle $\phi' = \tan^{-1} 0.30 = 16°40'$. Figure 208($b$) shows the block as a free body. Since slipping is impending, the reactions R_1 and R_2 are acting at the angle $\phi' = 16°40'$ with the normal to the surface of contact, or at 26°40' with the vertical. The force triangle is shown in Fig. 208(c), from which R_1 and R_2 scale 280 lb.

In Fig. 208(*d*) is shown the left wedge as a free body, with the known force R_1' equal and opposite to R_1 acting upon it. The unknown forces are P, horizontal, and R_3 acting to oppose motion at the angle $\phi' = 16°40'$ with the normal. The force triangle is shown in Fig. 208(*e*), from which P scales 200 lb., and R_3 scales 260 lb.

Algebraic Solution.—With the 500-lb. weight, Fig. 208(*b*), as the free body, equation $\Sigma F_x = 0$ gives

$$R_1 \sin 26°40' = R_2 \sin 26°40'$$
$$R_1 = R_2$$

Equation $\Sigma F_y = 0$ gives

$$2R_1 \cos 26°40' = 500$$
$$R_1 = 280 \text{ lb.}$$

With the left wedge, Fig. 208(*d*), as the free body, $\Sigma F_y = 0$ gives

$$280 \cos 26°40' = R_3 \cos 16°40'$$
$$R_3 = 261 \text{ lb.}$$

Equation $\Sigma F_x = 0$ gives

$$P = 280 \sin 26°40' + 261 \sin 16°40'$$
$$P = 126 + 75 = 201 \text{ lb.}$$

Trigonometric Solution.—In this problem, the trigonometric solution is easily applied. By the sine law, from Fig. 208(*c*),

$$R_1 = R_2 = 500 \times \frac{\sin 26°40'}{\sin 126°40'}$$
$$R_1 = R_2 = 500 \times \frac{0.4488}{0.802} = 280 \text{ lb.}$$

By the sine law, from Fig. 208(*e*),

$$P = 280 \times \frac{\sin 43°20'}{\sin 73°20'}$$
$$P = 280 \times \frac{0.6862}{0.9580} = 201 \text{ lb.}$$

If reaction R_3 is desired, it is given by the same triangle.

$$R_3 = 280 \times \frac{\sin 63°20'}{\sin 73°20'}$$
$$R_3 = 280 \times \frac{0.8936}{0.9580} = 261 \text{ lb.}$$

Problems

1. Solve for the horizontal forces necessary to pull the wedges out from under the block, Fig. 208(*a*), if the value of f is 0.4. *Ans.* 153 lb.

2. In Fig. 209, solve for the horizontal force P necessary to start the wedge to the right under the block, if the coefficient of friction is 0.2.

Ans. $P = 720$ lb.

3. In Fig. 209, solve for the horizontal force P necessary to start the wedge to the left, out from under the block. *Ans.* 133 lb. to the left.

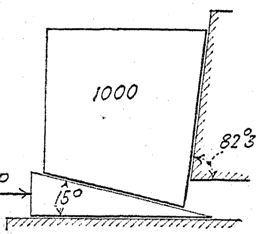


Fig. 209.

63. Laws of Friction.—The laws of friction for dry surfaces were deduced chiefly from the experiments of Morin, Coulomb, and Westinghouse. These may be stated as follows:

1. Friction varies directly as the normal pressure.
2. Limiting static friction is slightly greater than kinetic friction.
3. Ordinary changes of temperature affect friction only slightly.
4. At slow speeds, friction is independent of the speed. At high speeds, friction decreases as the speed increases, probably because of the fact that a film of air is drawn in and acts as a lubricant.
5. Kinetic friction decreases with the time.
6. Friction is increased by a reversal of motion.

The laws for lubricated surfaces are decidedly different from those for dry surfaces. For instance, friction is practically independent of the nature of the surfaces, owing to the fact that the chief friction is between the different layers of the lubricant. Limiting static friction is much greater than kinetic friction, as a result of the fact that while at rest the film of lubricant is pressed out from between the surfaces. Ordinary changes of temperature make a decided difference in the character of many lubricants and therefore affect the amount of friction greatly. Heavy normal pressure tends to force out the lubricant and therefore increases the coefficient of friction. As the lubrication becomes poor, the laws approach those for dry surfaces.

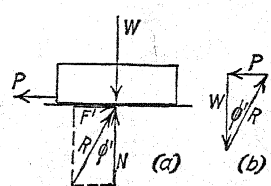


Fig. 210.

64. Least Pull and Cone of Friction.—If the force P, Fig. 210(a), acts horizontally on a body of weight W, and motion is impending, the force diagram is as shown in Fig. 210(b). If W and ϕ' are known, R and P can be determined, since for equilib-

rium the force polygon must close. If the force P is acting upward at the angle θ with the horizontal, as in Fig. 211(a), N is decreased, and therefore F' is decreased. Their ratio and the angle ϕ' remain constant. From the force diagram, Fig. 211(b), it is plain that with W constant and the direction of R constant, the minimum force P to close the force triangle must be acting at an angle of 90° with R. So with angle θ varying, the least pull P to start the block is given when $\theta = \phi'$.

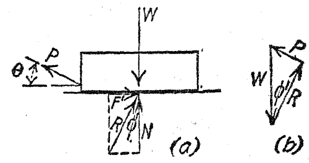


FIG. 211.

This result may also be obtained by means of the calculus method.

If P is acting downward at the angle θ with the horizontal, N and F' are increased, as will be seen in Fig. 212(a) and (b). If the body is free to move in any direction, the cone whose vertex is at A and whose axis is normal to the surface at A with angle of $2\ \phi'$ is called the *cone of friction*. If the resultant of P and W falls

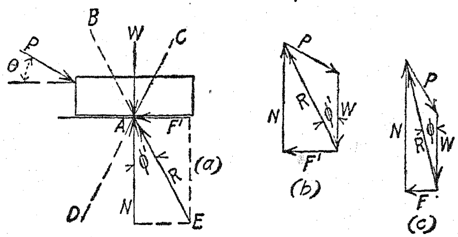


FIG. 212.

inside the cone of friction, it is evident that the reaction of the supporting plane falls within the angle DAE, that is, at an angle ϕ with the normal which is less than ϕ', as shown in Fig. 212(c). The required frictional resistance F is less than the limiting value $F' = N \tan \phi'$; hence the plane will hold the body in equilibrium no matter how much P is increased, if θ is also increased so that the resultant of W and P continues to fall inside the cone of friction.

Problems

1. A block weighing 100 lb. rests upon a plane surface inclined at an angle of 15° with the horizontal. If $f = 0.4$, what is the amount of the frictional force under the block? What force parallel to the plane is required to start the block down the plane? What horizontal force is necessary to start the block up the plane? *Ans.* 25.9 lb.; 12.7 lb.; 74.9 lb.

2. What is the least pull and its angle with the plane to start the block of Prob. 1 down the plane? Solve also for motion up the plane.

Ans. 11.9 lb., 21°50′; 59.95 lb., 21°50′.

3. In Prob. 1, how many degrees each side of the vertical is the angle for which no motion is possible, no matter how large a downward force P is applied?

Ans. 6°50′ above; 36°50′ below.

65. Friction on Square-threaded Screw.—In the case of the jackscrew and other square-threaded screws, the thread of the screw is an inclined plane wound around a cylinder. When the screw is used to lift a weight, the load acts vertically downward and is pushed up the inclined plane by means of a horizontal pressure applied at the end of a lever, as shown in Fig. 213. Let r be the mean radius of the thread, l the length of the lever, P the horizontal force at the end of the lever necessary to start motion, and Q the equivalent horizontal pressure at the mean radius of the thread. Then the value of Q is given by

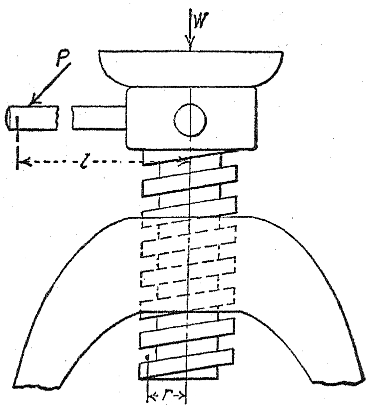


FIG. 213.

$$Q = \frac{Pl}{r}$$

Although the pressure of the weight W is distributed all around the circumference of the screw and over a number of threads, for simplicity it may be considered as though it were all acting on one small element, as shown in Fig. 214(a). In this figure, one circumference is shown developed. In Fig. 214(b), the element is shown as a free body, with the three forces acting upon it, W vertically downward, Q acting horizontally to push the element to the right up the plane, and R the reaction of the plane acting at the angle of friction ϕ' with the normal. Let p be the pitch of the thread. Then θ, the angle of inclination of the plane, is given by the expression

$$\theta = \tan^{-1} \frac{p}{2\pi r}$$

For motion impending up the plane, then,

$$Q = W \tan (\phi' + \theta)$$

as may be seen from the force diagram, Fig. 214(c).

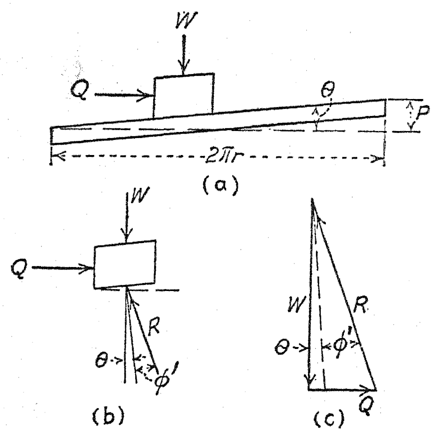


FIG. 214.

If motion down the plane is impending, the frictional resistance is reversed, and the reaction R is inclined oppositely, at the angle ϕ' with the normal, as shown in Fig. 215(b). For motion impending down the plane,

$$Q = W \tan (\phi' - \theta)$$

If the angle ϕ' is just equal to θ, the weight W will be just ready to slip; if ϕ' is less than θ, the weight W will slip down the plane unless held by a resistance Q. Jackscrews should always be constructed so that ϕ' is greater than θ.

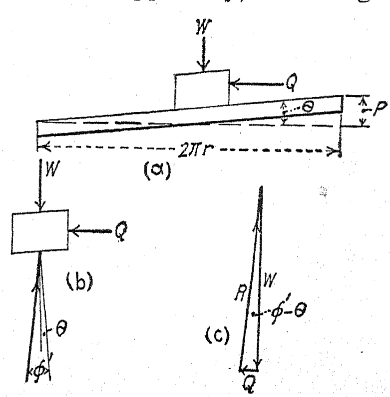


FIG. 215.

Problems

1. The pitch of a jackscrew is 0.5 in., the mean radius of the threads is 1.5 in., and the length of the lever is 2 ft. If $f = 0.08$, what force P will be necessary to start the jackscrew to raise a weight of 3 tons?

Ans. 50.1 lb.

2. In the jackscrew described in Prob. 1, what force P will be necessary to start it in the other direction? *Ans.* 10.1 lb.

66. Friction on Pivots and Ring Bearings.—A flat-end pivot and its bearing are originally perfect planes, but they cannot remain so after wear begins. The unit pressure at first is constant over the whole surface; but since the distance traveled over by any elementary area per revolution varies with its radial distance, the wear is greater at the outside. This reduces the pressure at the outside and increases t toward the middle. It is evident that after the pivot has run until conditions are uniform, the wear parallel to the axis on the pivot and bearing must be the same at all points. The wear on any unit area varies both with the distance traveled (or its radial distance) and with the normal pressure. Therefore, in order that the wear may be uniform over the whole area, it is necessary that the product of the normal pressure on any unit area and the radial distance of the area shall be constant. If p is the variable unit pressure, and ρ the distance of the unit area from the center, $p\rho$ must be constant, or

$$p\rho = K$$

Figure 216 represents a solid flat-end pivot, and Fig. 217 a hollow flat-end pivot. In either case, $dA = \rho\, d\rho\, d\theta$. The normal

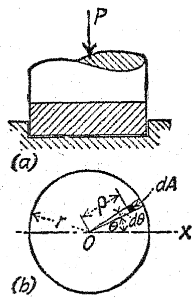


FIG. 216.

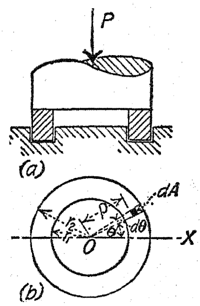


FIG. 217.

pressure on dA is $p\rho\, d\rho\, d\theta = K\, d\rho\, d\theta$. The frictional force on dA is $fK\, d\rho\, d\theta$, f being the coefficient of kinetic friction. The moment of this frictional force on dA about the center is $dM = fK\rho\, d\rho\, d\theta$. For the solid pivot of radius r, the total moment about the center is

$$M = fK\int_0^r \int_0^{2\pi} \rho\, d\rho\, d\theta = fKr^2\pi$$

As just given, the normal pressure on dA is $K\,d\rho\,d\theta$. The total normal pressure is

$$P = K\int_0^r\int_0^{2\pi} d\rho\,d\theta = K2\pi r$$

From this,

$$K = \frac{P}{2\pi r}$$

By substitution of this value in the expression for the moment,

$$M = fP\frac{r}{2}$$

This is seen to be a moment equivalent to the total frictional force fP acting at the mean radius $r/2$.

For the hollow pivot with inner radius r_1 and outer radius r_2, the total moment of the frictional force about the center is

$$M = fK\int_{r_1}^{r_2}\int_0^{2\pi} \rho\,d\rho\,d\theta = fK\pi(r_2^2 - r_1^2)$$

The normal pressure on dA is $K\,d\rho\,d\theta$. The total normal pressure is

$$P = K\int_1^{r_2}\int_0^{2\pi} d\rho\,d\theta = K2\pi(r_2 - r_1)$$

From this,

$$K = \frac{P}{2\pi(r_2 - r_1)}$$

By substitution of this value in the expression for the moment,

$$M = fP\left(\frac{r_2 + r_1}{2}\right)$$

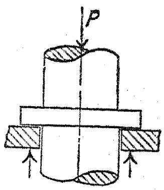


FIG. 218.

As before, this is seen to be a moment equivalent to the total frictional force fP acting at the mean radius $\left(\frac{r_2 + r_1}{2}\right)$.

The collar bearing, shown in Fig. 218, is the same as the hollow pivot. It has the advantage that it can be placed at any point along the shaft and also that several can be used on one shaft in order to obtain any desired amount of bearing area.

Conical Pivot.—Figure 219 represents a conical pivot under axial load P. Let dP be the load on area $dA = \rho\ d\rho\ d\theta$, and let dN be the normal pressure of the bearing on the slant area corresponding. Then, since $\Sigma F_y = 0$,

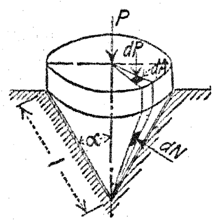


FIG. 219.

$$dN \sin \alpha = dP$$

or

$$dN = \frac{dP}{\sin \alpha}$$

The friction caused by the normal pressure dN is $f\ dN = f\ dP/\sin \alpha$, and its moment about the center is $dM = f\rho\ dP/\sin \alpha$. If p is the variable unit pressure on the cross-sectional area, since the same conditions hold true as in the flat-end pivot,

$$dP = p\rho\ d\rho\ d\theta = K\ d\rho\ d\theta$$

The total moment of the frictional forces about the center is

$$M = \int \frac{f\rho\ dP}{\sin \alpha} = \frac{fK}{\sin \alpha}\int_0^r \int_0^{2\pi} \rho\ d\rho\ d\theta = \frac{fK}{\sin \alpha}\pi r^2$$

As in the flat-end pivot,

$$K = \frac{P}{2\pi r}$$

so

$$M = \frac{fPr}{2 \sin \alpha}$$

Since $r/\sin \alpha = l$, the length of an element of the cone of contact, the expression for the moment becomes

$$M = fP\frac{l}{2}$$

It will be seen that the moment of the frictional force on a conical pivot is the same as that on a flat-end pivot whose radius is equal to the length of the element of the cone of contact.

Problems

1. A gin pole 16 in. in diameter at the base supports a total vertical load of 6000 lb. If the pole has a flat end at the bottom for which $f = 0.5$, what pressure at the end of a canthook 6 ft. long will be necessary to twist the pole about its own axis? *Ans.* 167 lb.

2. A propeller shaft 6 in. in diameter has six collar bearings, each 10 in. in diameter outside. If $f = 0.05$, and the shaft has an end thrust of 200,000 lb., what is the moment of the frictional force? What is the average unit pressure on the bearings? *Ans.* 3330 lb.-ft.; 665 lb./sq. in.

3. A conical pivot 3 in. in diameter with axis vertical for which $\alpha = 35°$ supports a vertical load of 6000 lb. If $f = 0.09$, what is the moment of the frictional resistance? *Ans.* 58.84 lb.-ft.

67. Axle Friction: Graphic Solution.—If a cylindrical axle of radius r rests in a bearing and is rotated, the axle will first roll from its position of rest until the resultant reaction of the bearing (resultant of N and F) acts at the angle of friction ϕ' with the radius at the point of contact, when slipping of the axle in the bearing takes place. The circle drawn concentric with the axle and tangent to the line of this reaction has a radius $r \sin \phi'$ and is called the *friction circle*. If the value of ϕ' is small, the sine of ϕ' may be taken as equal to the tangent of ϕ', or f. Then the radius of the friction circle is fr with very little error.

The radius r and the angle ϕ' are usually known, so the friction circle may be used to locate the point of contact of the axle and the bearing. Its chief use is in the graphic solution. In Fig. 220, Q is the resistance, and P is the working force. These intersect at B, so the resultant reaction of the bearing must also pass through B. Since this resultant reaction must also be tangent to the friction circle, the point of contact of the axle and bearing is determined.

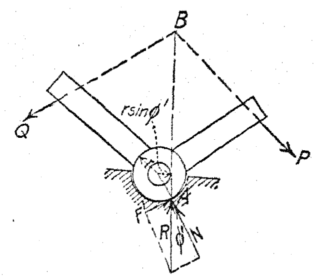


FIG. 220.

To determine the side of the friction circle at which the reaction is tangent, it is necessary to note the direction of pressure and the point of contact of the axle with the bearing. The reaction is tangent to the friction circle on that side toward which the axle rolls as it rotates.

A handy working method of determining this is as follows: On one of the members, place an arrow showing its action on the bearing. At right angles to this arrow, place a curved arrow showing the relative motion of the member around the bearing. Rotate the second arrow around the bearing until it agrees in direction with the first. It will then be on the side at which the reaction is tangent.

Another rule is that friction, being a resistance, always shortens the lever arm of the working force and lengthens the lever arm of the resisting force.

Problems

1. Figure 221 shows a simple steam hoist. Solve for the value of the force P necessary for uniform motion in the position shown, (1) if friction is neglected; (2) if friction is considered and $f = 0.15$ for all moving surfaces.

Ans. (1) $P = 3123$ lb.; (2) 3375 lb.

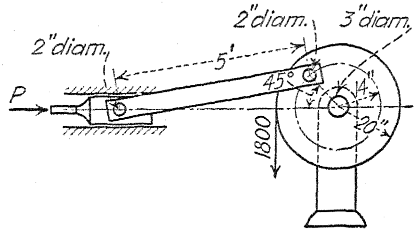


FIG. 221.

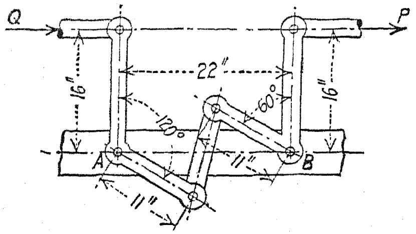


FIG. 222.

2. Figure 222 shows the standard compensator for interlocking signal systems in mean temperature position. Points A and B are fixed. All pins are 1 in. in diameter. Use $f = 0.10$, and determine the value of Q for $P = 50$ lb.

Ans. $Q = 49$ lb.

68. Friction of Flexible Belts and Bands.—If the belt shown in Fig. 223(a) is turning the pulley against some resistance, the tension T_2 on the driving side is greater than the tension T_1 on the slack side. Consider a piece of the belt of ds length as a free body, Fig. 223(b). Let dP be the normal pressure of the pulley on the belt on ds length. Since the free body is in equilibrium under the action of the forces shown, and since the thickness of the belt is usually small compared with the radius of the pulley, the equation $\Sigma M_0 = 0$ gives

$$r\,dF - r\,dT = 0$$
$$dF = dT$$

By summing forces in the radial direction,

$$dP = T \sin \frac{d\theta}{2} + (T + dT) \sin \frac{d\theta}{2} = 2T \sin \frac{d\theta}{2} + dT \sin \frac{d\theta}{2}$$

The term $dT \sin \frac{d\theta}{2}$ may be neglected, since it is a differential of a higher order, and $\sin \frac{d\theta}{2}$ may be replaced by $\frac{d\theta}{2}$. Then

$$dP = T\, d\theta$$

When slipping impends,

$$dF = f\, dP$$

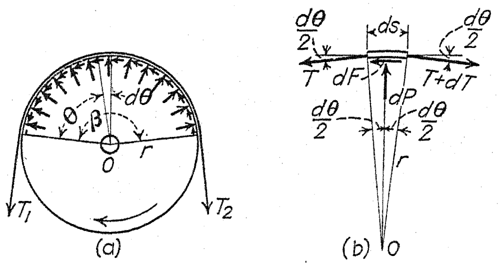


FIG. 223.

Therefore,

$$dT = f\, dP = fT\, d\theta$$

or

$$\int_{T_1}^{T_2} \frac{dT}{T} = f \int_0^{\beta} d\theta$$

By integration,

$$\log_e \frac{T_2}{T_1} = f\beta$$

In terms of common logarithms, this becomes

$$\log_{10} \frac{T_2}{T_1} = 0.4343 f\beta$$

In the exponential form it becomes

$$\frac{T_2}{T_1} = e^{f\beta}$$

The value of e, the base of the natural system of logarithms, is 2.71828.

The angle β is in radians. If the belt is slipping, the same relations hold true, f being the coefficient of kinetic friction. These relations hold true also for a rope around a spar or snubbing post and for a flexible band on a drum.

If a belt is moving around a pulley at high speed, the centrifugal action on the belt reduces the normal pressure and hence also the frictional driving force that may be developed.

It will be noticed that the equation for the relation between the tensions on the two sides of a pulley does not contain the dimension of the pulley, and therefore the tensions are independent of the size of the pulley.

These relations are not true if slipping is neither occurring nor impending.

Problems

1. A belt runs between two pulleys, one of which is 2 ft. in diameter, the other 6 in. in diameter, with their centers 22 in. apart. If the value of f for the larger pulley is 0.4, what must be the value of f for the smaller pulley so that slipping would impend on both pulleys at the same time?

Ans. 0.695.

2. A windlass has 3.5 turns of rope around the drum. The value of f is 0.36. If the load being pulled by the rope is 12,000 lb., what must be the tension on the other end of the rope to prevent slipping? *Ans.* 4.37 lb.

3. The tension in the free end of the rope of a block and tackle is 500 lb. It is held by being passed around a post for which $f = 0.5$. How many turns are required to hold it if the tension in the slack end is not to exceed 10 lb.? *Ans.* 1.25.

69. Summary of Principles of Friction.—In the solution of problems involving friction, several principles are to be noted particularly.

1. If friction is neglected, reactions are always normal to the surfaces.

2. If the free body is in motion or tends to move, the friction of adjoining surfaces upon the free body opposes its motion.

3. If the free body is at rest, and the adjoining surfaces move or tend to move over it, the friction upon the free body is in the direction of the moving surface.

4. The coefficient of static friction is used to determine the friction only when the body is at rest, *with slipping impending.*

When slipping is not impending, static conditions determine the friction.

70. Rolling Resistance.—If the curved surface of a perfect cylinder touches a perfect plane, they are in contact only along a line. If a loaded wheel rests upon a rail or roadway, a deformation is caused so that there is an area of contact. If a horizontal pull P, Fig. 224, is applied to the axle to move the wheel forward uniformly, the resultant reaction R of the supporting surface acts at a point B in front of the vertical radius. Let the horizon-

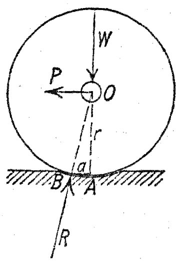


FIG. 224.

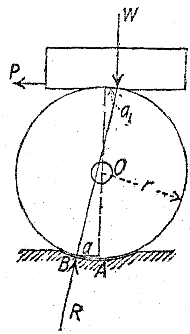


FIG. 225.

tal distance AB be called a. If motion is uniform and if the indentation is small, equation $\Sigma M_B = 0$ gives, approximately,

$$Pr = Wa$$
$$P = \frac{Wa}{r}$$

If the load W is applied at the circumference of the wheel or roller, as in Fig. 225, and a force P is applied to move both load and roller forward uniformly, a similar relation is obtained. Let a be the distance from the point of application of the resultant to the vertical radius at the bottom of the roller, and a_1 that at the top. Then the equation $\Sigma M_B = 0$ gives

$$2Pr = W(a_1 + a)$$

If $a_1 = a$,

$$P = \frac{Wa}{r}, \text{ as before.}$$

If the weight W is carried by two or more rollers, $R_1 + R_2 + \cdots = W$ (approx.). Equation $\Sigma F_x = 0$ gives

$$P = (R_1 + R_2 + \cdots) \sin \phi$$

if ϕ is the angle between R and the vertical. Then

$$P = W \sin \phi$$

$$P = W\frac{a_1 + a}{2r}$$

If $a_1 = a$,

$$P = \frac{Wa}{r}$$

Experiments appear to show that the distance a is practically constant for the same materials, both for varying loads and for varying radii, within reasonable limits. It is called the *coefficient of rolling friction*, or, preferably, the *coefficient of rolling resistance*. The experiments of Coulomb, Weisbach, and Pambour give the following values for a in inches.

Wheel	Track	a, inches
Elm	Oak	0.0327
Lignum vitae	Oak	0.0195
Cast iron	Cast iron	0.0183
Cast iron or steel	Steel	0.007 to 0.020

In roller bearings, use is made of the fact that hard steel rollers on hard steel have very little resistance. Figure 226 shows such a bearing. The axle a in rotating in bearing b rolls on the rollers c instead of sliding directly on the bearing b. If the pressure is light, balls may be used instead of rollers.

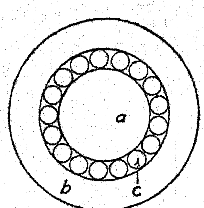


FIG. 226.

Problems

1. A 125,000-lb. freight car with 33-in. wheels and 4-in. axles for which $f = 0.07$ requires 1200 lb. drawbar pull to keep it in uniform motion on a level track. Compute the value of the coefficient of rolling resistance a.

Ans. $a = 0.0185$ in.

2. A cast-iron engine frame weighing 6400 lb. rests upon steel rollers 2 in. in diameter, which in turn rest upon pine timbers. If the coefficient of

rolling resistance for cast iron on steel is 0.018 in., and that for steel rollers on pine timber is 0.033 in., what horizontal force is necessary to move the frame at a uniform speed? *Ans.* 163.2 lb.

GENERAL PROBLEMS ON FRICTION

1. If the static coefficient of friction of cast-iron wheels on steel rails is 0.25, what is the limiting slope down which cars may be run at uniform speed? *Ans.* 14°02′.

2. If the kinetic coefficient of friction of cast-iron wheels on steel rails is 0.22, and the brakes are tightened so that the wheels skid on the rails, what is the unbalanced force down the limiting slope determined in Prob. 1 for a car weighing 40,000 lb.? *Ans.* 1162 lb.

3. In Fig. 227, a wedge with an angle of 20° is forced under a weight of 1000 lb. held against a vertical wall A. If the angle of friction ϕ' is 15° for all surfaces, what horizontal force P is necessary to start the wedge to the right? *Ans.* 1192 lb.

4. In Fig. 227, solve for the horizontal force acting toward the left to start the wedge out from under the block. *Ans.* 176 lb.

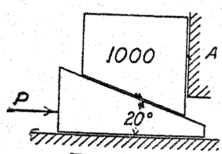


FIG. 227.

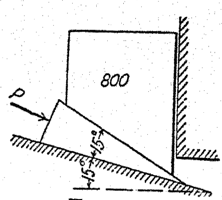


FIG. 228.

5. If the coefficient of friction $f = 0.25$ for all surfaces of the wedge and block (Fig. 228), solve for the force P to start the wedge to the right. The line of action of force P bisects the angle of the wedge. *Ans.* 1077 lb.

6. Solve for the horizontal force necessary to start a 200-lb. cake of ice up a wooden chute at an angle of 10° with the horizontal if $f = 0.05$. *Ans.* 45.7 lb.

7. Solve Prob. 6 for the least force necessary. *Ans.* 44.54 lb.

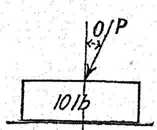


FIG. 229.

8. If the coefficient of friction $f = 0.4$ for the block shown in Fig. 229, and the angle $\theta = 25°$, what pressure P will be necessary to start the block? *Ans.* 66.5 lb.

9. A block of stone weighing 600 lb. rests on a plane at an angle of 30° with the horizontal for which $f = 0.75$. Compute the friction under the

block. Compute the minimum force necessary to start the block down the plane. *Ans.* 300 lb.; 71.8 lb.

10. The two blocks shown in Fig. 230 are connected by a cord passing over a pulley at C, the friction of which is neglected. Will the system move if the coefficient of friction under the 20-lb. block is 0.25 and that under the 50-lb. block is 0.30? Compute F_1, F_2, and T.

Ans. 4.33 lb.; 14.35 lb.; 5.67 lb.

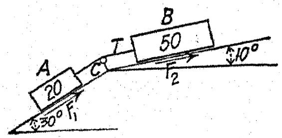


FIG. 230.

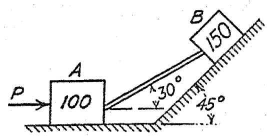


FIG. 231.

11. The bar between the two blocks shown in Fig. 231 can take compression. Neglecting the weight of the bar, solve for the horizontal force P necessary to start the blocks to the right if the coefficient of friction $f = 0.4$.

Ans. 223.4 lb.

12. In Fig. 231, determine whether or not the blocks will slide to the left if force P is removed. Solve for the compression in the bar and for the frictional force under each block. *Ans.* 59.6 lb.; $F_A = 51.7$ lb.; $F_B = 48.6$ lb.

13. Solve for the horizontal force P to start the blocks shown in Fig. 232 to the left if the angle of friction $\phi' = 15°$. *Ans.* 246 lb.

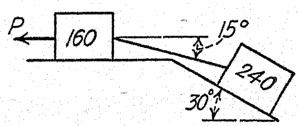


FIG. 232.

14. Solve for the amount and direction of the minimum force P that will start the blocks shown in Fig. 232 to the left if $f = 0.3$.

Ans. 252 lb. at 16°40′ with H.

15. A uniform ladder 20 ft. long weighing 60 lb. is placed with its lower end on a horizontal floor and leaned against a vertical wall at an angle of 10° with the wall. The coefficient of friction at the floor is 0.3, and that at the wall is 0.2. What horizontal force applied at a point 4 ft. up along the ladder will cause slipping outward? What horizontal force at the same point will cause slipping inward? *Ans.* 15.8 lb.; 30.5 lb.

16. If the bottom of the ladder described in Prob. 15 is moved farther from the wall, what is the maximum angle with the wall at which it can be placed before slipping impends? *Ans.* 32°30′.

17. If for the ladder described in Prob. 15 the coefficient of friction at both floor and wall is 0.14, how far up the ladder may a weight of 160 lb. be placed before slipping impends? *Ans.* 18.2 ft.

18. What is the minimum value of f for the ladder described in Prob. 15 that will prevent slipping if a weight of 160 lb. is placed 4 ft. from the upper end? *Ans.* 0.126.

19. If the coefficient of friction $f = 0.2$ for the hanger AB, Fig. 233, which slides up and down on the post MN, determine how close to the post the load P may be placed without causing the hanger to slide down. Neglect the weight of the hanger. *Ans.* 3.5 in.

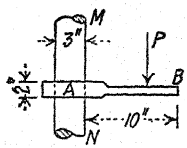


FIG. 233.

20. If the coefficient of rolling resistance $a = 0.015$ in. for the wheels of a freight car on steel rails, and $f = 0.036$ for axle friction, determine the horizontal drawbar pull necessary to keep a car weighing 160,000 lb. in uniform motion on a level track. The wheels are 33 in. in diameter, and the axles are 4 in. in diameter. Determine also the steepest grade on which the car would not start if brakes are not applied.

Ans. 844 lb.; 0.528 per cent.

21. A collar bearing 16 in. in diameter on a 12-in. shaft carries a thrust of 8000 lb. If $f = 0.04$, what is the moment of the frictional force and how much work is lost in friction per revolution? *Ans.* 187 lb.-ft.; 1173 ft.-lb.

22. A weight of 10,000 lb. is being lowered into the hold of a ship by a rope passing around a spar for which $f = 0.2$. If the resistance at the other end of the rope is not to exceed 150 lb., how many turns of rope around the spar are necessary? *Ans.* 3.34.

23. A rope has $3\frac{1}{2}$ turns around a windlass for which $f = 0.25$. If the pull necessary to keep the rope from slipping is 50 lb., what pull is being exerted at the other end? *Ans.* 12,180 lb.

24. A rope has three turns around a post for which $f = 0.3$. If the pull on one end of the rope is 60,000 lb., what is the minimum pull on the other end that will just hold it? *Ans.* 210 lb.

25. With $2\frac{1}{2}$ turns of cord around a bar, a pull of 1 lb. held a weight of 30 lb. Compute the value of f. *Ans.* 0.216.

CHAPTER IX

CENTROIDS AND CENTERS OF GRAVITY

71. Centroid of a System of Forces with Fixed Application Points.—In all the previous discussions of forces applied to rigid bodies, it has been assumed that the force could be applied at any point along its line of action. In some cases forces are considered to be applied at certain definite points which remain fixed, no matter how the body is displaced or the system of forces rotated. Consider a system of particles each of which is acted upon by a force proportional to its mass, and let these forces be parallel to each other. It is evident that if the system of particles is rotated while the forces remain fixed in direction, the result is the same as if the system remained fixed in space and the force system were rotated, each force about its point of application.

Let such a force system be acting upon a system of particles in the direction of the Y axis. The distance of the resultant from the XY plane and also from the YZ plane may be determined by the theorem of moments. Then consider each force of the system to be rotated about its point of application until the system of forces is parallel to the X axis. The line of action of the resultant is necessarily at the same distance from the XY plane that it was before rotation. Also, its distance from the XZ plane may now be determined, and its point of intersection with the line of action of the resultant in its original position must necessarily be the point about which the resultant was rotated. Next, if from this position each force is rotated about its point of application until it is parallel to the Z axis, the line of action of the resultant is necessarily at a fixed distance from the XZ plane during the rotation.

Finally, if from this last position each force is rotated about its point of application back to its original position parallel to the Y axis, the line of action of the resultant remains at a fixed distance from the YZ plane and must necessarily return to its original position. In order for it to do this, the last two rotations must necessarily have been made about the same point as the first.

For if the second rotation had been made about a point on the resultant that had a different X coordinate from the first point of rotation, the final position of the resultant would have had a different X coordinate and therefore could not have coincided with the original position. Similarly, if the third rotation had been made about a point that had a different Z coordinate from the first point of rotation, the final position of the resultant would have had a different Z coordinate and therefore could not have coincided with the original position. This point in the resultant is therefore the *one fixed point* in the system for any possible rotation and is called the *centroid* of the system. Its coordinates are denoted by $\bar{x}$, $\bar{y}$, $\bar{z}$ (called *gravity x, gravity y, gravity z*).

Each particle of a body is attracted by the earth, and the force of this attraction is proportional to the mass of the particle. It is obvious that the points of application of these forces remain unchanged for all positions of the body and that the lines of action of the forces for bodies of the size considered in engineering problems are practically parallel. The resultant of all these attractive forces is called the *force of gravity* or the *weight* of the body, and its fixed application point is called the *center of mass* or *center of gravity* of the body. Ordinarily it is necessary only to consider this resultant force.

In case the application points of the forces of a system are fixed and coplanar, two moment equations will be sufficient to locate the centroid if the axes are taken in the plane of the application points. Since, as just discussed above, the centroid remains fixed with respect to the system during any rotation, the forces may be assumed to be rotated until they are normal to the plane of the application points. Then, by the theorem of moments,

$$\bar{x} = \frac{\Sigma Fx}{\Sigma F}$$

$$\bar{y} = \frac{\Sigma Fy}{\Sigma F}$$

Problems

1. Four parallel forces have amounts and application points in the XY plane as follows: 10 lb. (0″, 0″); 16 lb. (12″, 4″); 30 lb. (3″, 5″); 35 lb. (4″, 10″). Locate the centroid if all the forces are in the same direction.
Ans. 4.64″, 6.20″.

2. Solve Prob. 1 if the first two forces are reversed in direction.
Ans. 0.97″, 11.18″.

72. Centroids of Solids, Surfaces, and Lines Defined.—The centroid of a geometric solid is that point which coincides with the center of mass of a homogeneous body occupying the same volume.

The centroid of a surface is the limiting position of the center of gravity of a homogeneous thin plate, one face of which coincides with the surface as the thickness of the plate approaches zero.

The centroid of a line is the limiting position of the center of gravity of a homogeneous thin rod whose axis coincides with the line as the cross-sectional area of the rod approaches zero.

73. Moment with Respect to a Plane.—The moment of a force with respect to a plane parallel to its line of action is the product of the force and the perpendicular distance from the force to the plane, as discussed in Art. 53. By analogy, the moment of a solid, surface, or line with respect to a plane is equal to the product of the solid, surface, or line and the perpendicular distance from the plane to its centroid. Since solids, surfaces, and lines are not vector quantities, the sign of the moment must be provided for by assigning the plus sign to the ordinates on one side of the plane and the minus sign to those on the other.

By the principle of Art. 53, the moment of the weight of a body with respect to a plane is equal to the sum of the moments of the weights of the several particles of the body with respect to the same plane.

For the ZY plane,

$$W\bar{x} = \Sigma wx$$

For the XY plane,

$$W\bar{z} = \Sigma wz$$

For the XZ plane,

$$W\bar{y} = \Sigma wy$$

If the moment of the weight of a body with respect to a plane is zero, the center of gravity of the body is in that plane.

The moment of a solid, surface, or line with respect to a plane is equal to the moment of its separate component parts with respect to the same plane. For if w is the unit weight of a homogeneous body, and V is its volume, its total weight is $wV = W$. By the foregoing principle,

$$wV\bar{x} = wv_1x_1 + wv_2x_2 + wv_3x_3 + \cdots$$

or

$$V\bar{x} = v_1x_1 + v_2x_2 + v_3x_3 + \cdots$$

Similar propositions hold true for surfaces and lines.

If the moment of a solid, surface, or line with respect to a plane is zero, the centroid is in that plane; and, conversely, if the centroid of a solid, surface, or line is in a certain plane of reference, the moment with respect to that plane is zero.

74. Location of Centroids by Planes and Axes of Symmetry.— If a solid, surface, or line is symmetrical with respect to any plane, the centroid is in that plane.

If two or more planes of symmetry intersect in a line, this line is called an *axis of symmetry* and contains the centroid.

If three or more planes of symmetry intersect each other in a point, this point is the centroid.

Similar propositions are true for the center of gravity of a mass if homogeneous.

An observation of the planes of symmetry will enable the centroids of many geometrical figures to be located either partially or completely. The following are illustrations:

The centroid of a straight line is at its middle point.

The centroid of a circular arc, sector, or segment is on its bisecting radius.

The centroid of a circle or its circumference is the center of the circle.

The centroid of a rectangle or its perimeter is the intersection of the lines bisecting the pairs of opposite sides. It is likewise the intersection of the two diagonals, although in general these are not axes of symmetry.

The centroid of a sphere or of its surface is the center of the sphere.

The centroid of a cylinder or of its surface is the middle point of its axis.

The centroid of a right prism with parallel bases is the middle point of its axis.

The centroid of a right cone is on its axis.

The centroid of a thin plate is midway between the positions of the centroids of the faces.

75. Centroids of Some Simple Surfaces and Solids.—For many simple surfaces and solids, enough planes or lines containing the centroid may be determined to locate the centroid completely.

Triangle.—The centroid of a triangle is at the intersection of its medians. In Fig. 234, the centroid of any elementary strip MN parallel to the base and of infinitesimal width is on the median AD; therefore the centroid of the triangle is on the median. Likewise it is on the median BE and therefore is at their point of intersection O.

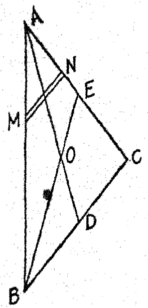


FIG. 234.

By geometry, $OD = \frac{1}{3}AD$. Therefore, the centroid is on any median, at a distance of one-third its length from its intersection with the base.

The perpendicular distance from O to BC is one-third the altitude of the triangle; therefore, the centroid of a triangle is at the intersection of two lines drawn parallel, respectively, to two sides of the triangle and distant one-third of the altitude from the base.

Slant Area of Pyramid.—The centroid of the slant area of a pyramid is on the axis of the surface, at a distance from the base equal to one-third the altitude. Consider the pyramid to be cut by planes parallel to the base and infinitesimal distances apart. The centroid of each infinitesimal area intercepted between two succeeding planes is on the axis; therefore, the centroid of the total area is on the axis. The centroid of each of the triangular faces is in a plane distant one-third the altitude from the base. Hence the centroid of the entire slant area is at the intersection of the axis with this plane.

Since the surface of a cone may be considered as the limit of the surface of a pyramid the number of whose sides is increased to infinity, the same proposition holds true for a cone.

Oblique Prism.—The centroid of an oblique prism with parallel bases is at the middle point of its axis. Consider the prism to be cut into elementary plates parallel to the base. The centroid of each plate approaches coincidence with the centroid of its area as its thickness approaches zero. The straight line joining these centroids is the axis of the prism by definition; hence the centroid of the prism is on its axis. Again, consider the prism to be made up of elementary rods parallel to the axis. The centroid of each rod is at its middle point; hence the centroid of

the prism is in the plane passed through these middle points of the elementary rods parallel to the base.

Oblique Pyramid or Cone.—The centroid of an oblique pyramid or cone is on its axis. Consider the pyramid or cone to be cut into elementary plates parallel to the base. The centroid of each plate approaches coincidence with the centroid of its area as its thickness approaches zero. The surface of each plate is an area similar to the area of the base, and its centroid is at a corresponding point in its area, hence on the axis. Therefore, since the centroids of all the elementary plates lie upon the axis, the centroid of the entire pyramid or cone is on the axis.

76. Centroids by Integration. *Lines, Plane Surfaces, and Solids.*—If a solid, surface, or line is divided into its infinitesimal parts, the principle of moments (Art. 53) may be stated as follows: For line of length l,

$$l\bar{x} = \int x\,dl;\ l\bar{y} = \int y\,dl;\ l\bar{z} = \int z\,dl \tag{1}$$

For surface of area A,

$$A\bar{x} = \int x\,dA;\ A\bar{y} = \int y\,dA;\ A\bar{z} = \int z\,dA \tag{2}$$

For solid of volume V,

$$V\bar{x} = \int x\,dV;\ V\bar{y} = \int y\,dV;\ V\bar{z} = \int z\,dV \tag{3}$$

These expressions may be used when any given solid, surface, or line cannot be divided into finite component parts whose centroids are known but is of such form that the differential expression for the moment can be integrated.

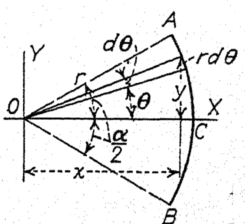


FIG. 235.

EXAMPLE 1

Locate the centroid of a circular arc.

Solution.—By symmetry, the centroid is on the axis OC, Fig. 235, so $\bar{y} = 0$. To determine $\bar{x}$, use expression (1).

$$l\bar{x} = \int x\,dl$$

$$l = r\alpha;\ x = r\cos\theta;\ dl = r\,d\theta$$

$$r\alpha\bar{x} = \int_{-\frac{\alpha}{2}}^{+\frac{\alpha}{2}} r^2 \cos\theta\,d\theta$$

$$r\alpha\bar{x} = r^2 \sin \theta \Big]_{-\frac{\alpha}{2}}^{+\frac{\alpha}{2}}$$

$$r\alpha\bar{x} = 2r^2 \sin \frac{\alpha}{2}$$

$$\bar{x} = \frac{2r}{\alpha} \sin \frac{\alpha}{2}$$

For $\alpha = 90°$,

$$\sin \frac{\alpha}{2} = 0.707; \bar{x} = 0.707 \frac{4r}{\pi} = 0.901r$$

For $\alpha = 180°$,

$$\sin \frac{\alpha}{2} = 1; \qquad \bar{x} = \frac{2r}{\pi} = 0.637r$$

EXAMPLE 2

Locate the centroid of the sector of a circle.

Solution.—Let the X axis bisect the angle of the sector (Fig. 236). Then

$$\bar{y} = 0$$

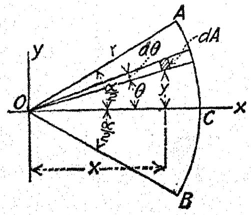


FIG. 236.

To determine $\bar{x}$, use expression (2).

$$A\bar{x} = \int x\, dA$$

$$A = \tfrac{1}{2}r^2\alpha;\; dA = \rho\, d\rho\, d\theta;\; x = \rho \cos \theta$$

$$\tfrac{1}{2}r^2\alpha\bar{x} = \int_0^r \int_{-\frac{\alpha}{2}}^{+\frac{\alpha}{2}} \rho^2 \cos \theta\, d\rho\, d\theta$$

$$\bar{x} = \frac{4r}{3\alpha} \sin \frac{\alpha}{2}$$

For $\alpha = 90°$,

$$\sin \frac{\alpha}{2} = 0.707; \bar{x} = 0.707 \frac{8r}{3\pi} = 0.6r$$

For $\alpha = 180°$,

$$\sin \frac{\alpha}{2} = 1; \bar{x} = \frac{4r}{3\pi} = 0.425r$$

The distance of the centroid of a quadrant from either bounding radius is likewise $4r/3\pi$.

EXAMPLE 3

Locate the centroid of a pyramid or cone.

Solution.—By Art. 75, the centroid is on the axis, so it remains to determine its distance from a plane through the vertex parallel to the base. Let the pyramid or cone be placed with its vertex at the origin of coordinates,

and let its base be normal to the X axis. Figure 237 shows a section through the pyramid or cone in the XY plane.

Let A be the area of the base, and let a be the area of any cross section parallel to the base at distance x from the vertex. Use expression (3).

$$V\bar{x} = \int x\,dV$$

$$V = \frac{Ah}{3};\ dV = a\,dx$$

$$\frac{Ah}{3}\bar{x} = \int xa\,dx$$

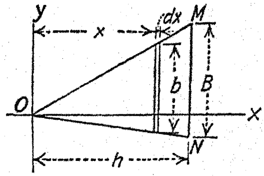


FIG. 237.

By similar triangles,

$$\frac{b}{B} = \frac{x}{h}$$

Also, by the geometry of similar areas,

$$\frac{a}{A} = \frac{b^2}{B^2} = \frac{x^2}{h^2}$$

Then

$$a = \frac{A}{h^2}x^2$$

$$\frac{Ah}{3}\bar{x} = \frac{A}{h^2}\int_0^h x^3\,dx$$

$$\bar{x} = \tfrac{3}{4}h$$

The distance of the centroid from the base is $\frac{1}{4}h$.

EXAMPLE 4

Locate the centroid of a hemisphere.

Solution.—Let the axes be placed as shown in Fig. 238. By symmetry, $\bar{y} = 0$ and $\bar{z} = 0$. To determine $\bar{x}$, use expression (3).

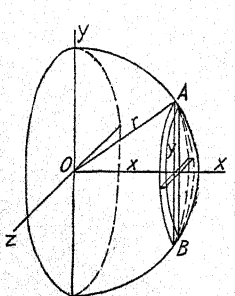


FIG. 238.

$$V\bar{x} = \int x\,dV$$

$$V = \tfrac{2}{3}\pi r^3;\ dV = \text{volume of slice } AB = \pi y^2\,dx$$

$$\tfrac{2}{3}\pi r^3\bar{x} = \int x\pi y^2\,dx$$

$$y^2 = r^2 - x^2$$

$$\tfrac{2}{3}r^3\bar{x} = \int_0^r r^2x\,dx - \int_0^r x^3\,dx$$

$$\tfrac{2}{3}r^3\bar{x} = \frac{r^4}{2} - \frac{r^4}{4}$$

$$\bar{x} = \tfrac{3}{8}r$$

Problems

1. Determine by integration the distance of the centroid of an arc of 90° from the radius at its end. *Ans.* $2r/\pi$.

2. Solve Example 2 by using as dA the differential sector OL, Fig. 239.

3. Solve Example 2 by using as dA the differential area MN, Fig. 239.

4. Determine by integration the distance of the centroid of a quadrant from the limiting radius.

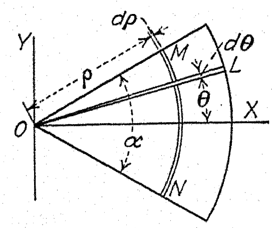


FIG. 239.

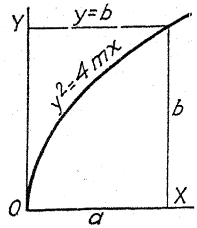


FIG. 240.

5. Locate the centroid of the parabolic half segment shown in Fig. 240.
Ans. $\bar{x} = 3a/5$; $\bar{y} = 3b/8$.

6. In Fig. 240, locate the centroid of the area between the parabola, the Y axis, and the line $y = b$. *Ans.* $\bar{x} = 3a/10$; $\bar{y} = 3b/4$.

77. Centroids of Surfaces and Solids of Revolution.—The centroid of a surface of revolution generated by the rotation of a line about an axis in its plane is on the axis. In determining its position on the axis, the solution may be simplified by using for dA the area generated by the length ds of the generating line as shown by the shaded part in the two views in Fig. 241.

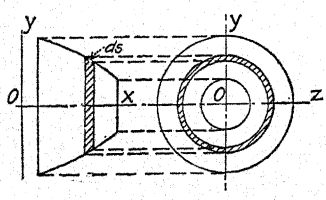


FIG. 241.

The centroid of a solid of revolution generated by the rotation of an area about an axis in its plane is on the axis. In determining its position on the axis, the solution may be simplified by using as dV the volume generated by the element dA of the generating area.

EXAMPLE 1

Locate the centroid of a hemispherical surface.

Solution.—Let the axes be placed as shown in Fig. 242. By symmetry, $\bar{y} = 0$ and $\bar{z} = 0$.

$$A = 2\pi r^2;\ dA = 2\pi y\ ds;\ x = r \cos \theta$$

$$A\bar{x} = \int x\ dA$$

$$y = r \sin \theta;\ ds = r\ d\theta$$

$$2\pi r^2\bar{x} = \int_0^{\frac{\pi}{2}} 2\pi r^3 \cos\theta \sin\theta \, d\theta$$

$$\bar{x} = \frac{r}{2}\sin^2\theta\Big]_0^{\frac{\pi}{2}}$$

$$\bar{x} = \frac{r}{2}$$

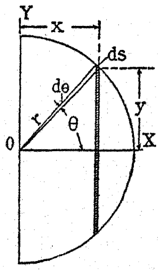


FIG. 242.

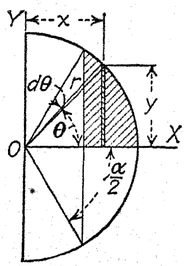


FIG. 243.

EXAMPLE 2

Locate the centroid of a spherical segment.

Solution.—Let the axes be placed as shown in Fig. 243. By symmetry $\bar{y} = 0$ and $\bar{z} = 0$.

$$V\bar{x} = \int x \, dV$$

$$dV = \pi y^2 \, dx; \; V = \int \pi y^2 \, dx; \; x = r\cos\theta$$

$$dx = -r\sin\theta \, d\theta; \; y = r\sin\theta$$

$$-\bar{x}\int_0^{\frac{\alpha}{2}} \pi r^3 \sin^3\theta \, d\theta = -\int_0^{\frac{\alpha}{2}} \pi r^4 \cos\theta \sin^3\theta \, d\theta$$

$$\bar{x}\int_0^{\frac{\alpha}{2}} \sin^3\theta \, d\theta = r\int_0^{\frac{\alpha}{2}} \sin^3\theta \cos\theta \, d\theta$$

$$-\frac{\bar{x}}{3}\Big[\sin^2\theta\cos\theta + 2\cos\theta\Big]_0^{\frac{\alpha}{2}} = \frac{r}{4}\Big[\sin^4\theta\Big]_0^{\frac{\alpha}{2}}$$

$$\bar{x} = \frac{3}{4}r\frac{\sin^4\frac{\alpha}{2}}{2 - 3\cos\frac{\alpha}{2} + \cos^3\frac{\alpha}{2}}$$

Problems

1. Show by integration that the centroid of the curved surface of a right circular cone is distant one-third the altitude from the base.

2. Locate the centroid of the frustum of a cone that has dimensions as shown in Fig. 244. (Consider the frustum as generated by the revolution of the shaded trapezoid about the X axis.) *Ans.* $\bar{x} = 4.184$ in.

78. Theorems of Pappus and Guldinus.

I. *The area S of the surface generated by any plane curve revolved about a nonintersecting axis in its plane is equal to the product of the length of the curve and the length of the path traced by the centroid of the curve.*

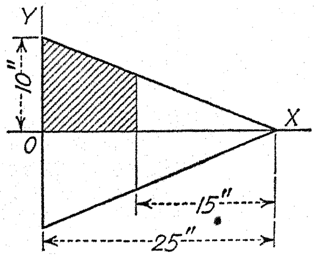


FIG. 244.

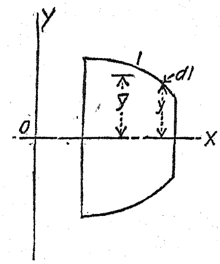


FIG. 245.

In Fig. 245, let l be the length of the generating curve, and dl a differential length of the curve at distance y from the axis of rotation OX. The area of the differential surface generated by dl as it rotates about axis OX is $2\pi y\, dl$. The area of the surface S is given by the expression

$$S = 2\pi \int y\, dl = 2\pi \bar{y} l$$

II. *The volume V of the solid of revolution generated by a plane area revolved about a nonintersecting axis in its plane is equal to the product of the area and the length of the path traced by the centroid of the area.*

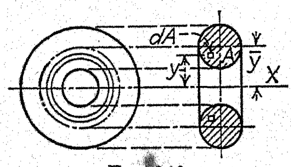


FIG. 246.

In Fig. 246, let A be the generating area, and dA the differential area at distance y from the X axis. The volume of the differential ring generated by dA as it rotates about the X axis is $2\pi y\, dA$. The volume V of the total solid of revolution is given by the expression

$$V = 2\pi \int y\, dA = 2\pi \bar{y} A$$

EXAMPLE 1

By means of Theorem I, determine the surface of a sphere.

Solution.—The surface of a sphere is generated by the rotation of a semicircular arc about the diameter through its ends, as ABC, Fig. 247. The

length of the arc is πr, and the distance from the X axis to its centroid is $2r/\pi$.

$$
\begin{aligned}
S &= 2\pi \bar{y} l \\
&= 2\pi \times \frac{2r}{\pi} \times \pi r \\
&= 4\pi r^2
\end{aligned}
$$

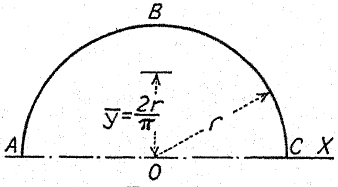


FIG. 247.

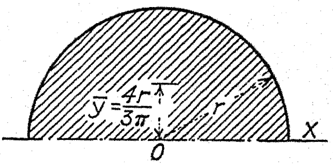


FIG. 248.

EXAMPLE 2

By means of Theorem II, determine the volume of a sphere.

Solution.—The volume of a sphere is generated by the rotation of a semicircular area about its bounding diameter, as shown in Fig. 248. The area is $\pi r^2/2$, and the distance of its centroid from the center is $4r/3\pi$

$$
\begin{aligned}
V &= 2\pi \bar{y} A \\
&= 2\pi \times \frac{4r}{3\pi} \times \frac{\pi r^2}{2} \\
&= \frac{4}{3}\pi r^3
\end{aligned}
$$

Problems

1. By means of Theorem I, determine the area of the base of a cone of radius r, altitude h, and slant height l. Determine also the area of the curved surface.

Ans. πr^2; $\pi r l$.

2. By means of Theorem II, determine the volume of a cone. *Ans.* $\pi r^2 h/3$.

3. The volume of the ellipsoid generated by rotating one-half the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ about the X axis is $\frac{4}{3}\pi ab^2$. Determine $\bar{y}$ by Theorem II. *Ans.* $4b/3\pi$.

4. Determine the volume of the solid of revolution generated by rotating the shaded area ABC, Fig. 249, about the X axis.

Ans. 113.1 cu. in.

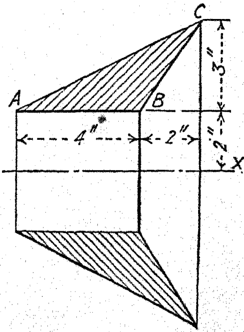


FIG. 249.

79. Centroids of Composite Lines, Surfaces, and Solids.—If a composite line, surface, or solid is made up of several simple parts whose centroids are known, the principle of Art. 53 may be applied.

The moment of a composite line, surface, or solid with respect to a plane is equal to the sum of the moments of the several component parts with respect to the same plane.

$$l\bar{x} = l_1x_1 + l_2x_2 + l_3x_3 + \cdots$$
$$A\bar{x} = A_1x_1 + A_2x_2 + A_3x_3 + \cdots$$
$$V\bar{x} = V_1x_1 + V_2x_2 + V_3x_3 + \cdots$$

Similar propositions hold true for $\bar{y}$ and $\bar{z}$.

If the line, surface, or solid was originally of simple form with one or more simple parts taken away, the equation is slightly modified. If l_1 is the line remaining after parts l_2, l_3, etc., have been taken away from the original line l,

$$l_1x_1 = l\bar{x} - l_2x_2 - l_3x_3 - \cdots$$

Similar propositions hold true for surfaces and solids.

The moment of a part of a line, surface, or solid with respect to a plane is equal to the moment of the entire line, surface, or solid minus the moment of the parts taken away.

EXAMPLE 1

Locate the centroid of the three lines shown in Fig. 250.

Solution.

$$12\bar{x} = (3 \times 0) + (4 \times 2) + (5 \times 5.25)$$
$$\bar{x} = 2.85 \text{ in.}$$
$$12\bar{y} = (3 \times 1.5) + (4 \times 0) + (5 \times 2.165)$$
$$\bar{y} = 1.28 \text{ in.}$$

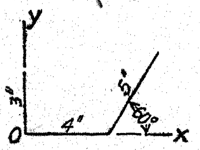


FIG. 250.

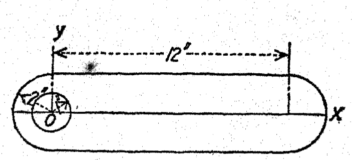


FIG. 251.

EXAMPLE 2

Locate the centroid of the area shown in Fig. 251.

Solution.—The center of the hole will be taken as the origin. By symmetry, $\bar{y} = 0$. Total original area = 60.56 sq. in. Area of hole = 3.14 sq. in. Remaining area = 57.42 sq. in.

$$57.42\bar{x} = (60.56 \times 6) - (3.14 \times 0)$$
$$\bar{x} = 6.34 \text{ in.}$$

Problems

1. Locate the centroid of the lines shown in Fig. 250 if the 3-in. part extends forward at right angles to OY and the 5-in. part is parallel to the Y axis. *Ans.* $\bar{x} = 2.33$ in.; $\bar{y} = 1.04$ in.; $\bar{z} = 0.375$ in.

2. Locate the centroid of the area shown in Fig. 251 if the part on the left of the Y axis is bent forward into the YZ plane.

Ans. $\bar{x} = 6.42$ in.; $\bar{z} = 0.081$ in.

3. Locate the centroid of the volume shown in Fig. 252, which consists of a parallelepiped, two cylinders, and a hemisphere. *Ans.* 14.18 in.

4. Locate the centroid of the volume shown in Fig. 252 if a cone is cut out of the solid. The base of the cone is in the YZ plane, and its axis is along the X axis. The radius of the base of the cone is 8 in., and the height of the cone is 30 in. *Ans.* 17.04 in.

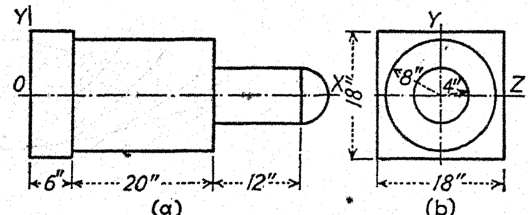


FIG. 252.

80. Center of Gravity of Composite Weights.—The principle of Art. 79 applies also to homogeneous bodies, the center of gravity of the body coinciding with the centroid of the volume. However, if a body is composed of several parts having different unit weights, each volume must be multiplied by the corresponding unit weight, and these weights used in the moment equation. In general, the center of gravity of such a composite weight does not coincide with the centroid of the volume.

Problems

1. Locate the center of gravity of the body shown in Fig. 252 if the parallelepiped and the central cylinder 32 in. long and 4 in. radius are made of steel, and the remainder of the larger cylinder and the hemisphere are made of brass. Steel weighs 490 lb./cu. ft., and brass weighs 534 lb./cu. ft. *Ans.* 14.3 in.

2. A steel cylinder is 4 in. in diameter and 2 ft. long. A concentric cylindrical hole 3 in. in diameter is to be bored into the cylinder at one end and filled with lead (706 lb./cu. ft.) so as to move the center of gravity ½ in. from its original position. Compute the depth of the cylindrical hole and the weight of lead required.

Ans. 5.56 in., 16.09 lb.; or 17.45 in., 50.35 lb.

81. Center of Gravity by Experiment.—The center of gravity of an irregular body may be determined by experiment. If the body is suspended freely, two intersecting vertical planes through the center of suspension may be marked on the body. Each of

these planes contains the center of gravity, therefore it is in their line of intersection. If the body is then suspended in some other position, the intersection of any other vertical plane through the center of support with the line of intersection of the other two planes determines the center of gravity.

If a body is of such form that it can easily be balanced across a knife edge, the position of the center of gravity may be determined readily. The body should be balanced perfectly in some position and the line of the supporting knife edge marked. The body should then be rotated and balanced and another line of support marked. The center of gravity is vertically above the intersection of the two lines.

GENERAL PROBLEMS ON CENTROIDS AND CENTERS OF GRAVITY

1. An endless wire, originally in the form of a circle 20 in. in diameter, is bent into the form of a semicircular arc and its diameter, as shown in Fig. 253. Locate the center of gravity. *Ans.* $\bar{x} = 4.75$ in.

2. Locate the center of gravity of the wire described in Prob. 1 if it is cut at point O, the upper part OA being bent around point A clockwise through an angle of 60° in the XY plane, and the lower part OB bent forward about point B until it is parallel to the Z axis.

Ans. $\bar{x} = 3.72$ in.; $\bar{y} = -0.595$ in.; $\bar{z} = 1.19$ in.

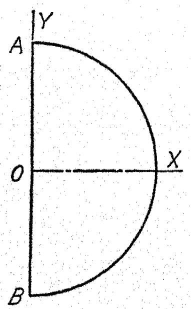


FIG. 253.

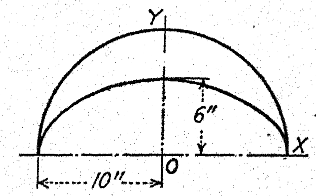


FIG. 254.

3. Locate the centroid of the area of one quadrant of an ellipse, the equation of which is $x^2 + 2.25y^2 = 9$.

Ans. $\bar{x} = 1.274$ in.; $\bar{y} = 0.849$ in.

4. Locate the centroid of the area shown in Fig. 254. The upper curve is a semicircle with a radius of 10 in. The lower curve is one-half of an ellipse.

Ans. $\bar{y} = 6.8$ in.

5. Locate the centroid of the area shown in Fig. 255.

Ans. $\bar{y} = 6.58$ in.

6. Locate the centroid of the area shown in Fig. 256.

Ans. $\bar{y} = 4.66$ in.

7. Locate the center of gravity of the trapezoidal-shaped piece of sheet metal shown in Fig. 257. *Ans.* $\bar{x} = 3.84$ in.; $\bar{y} = 2.19$ in.

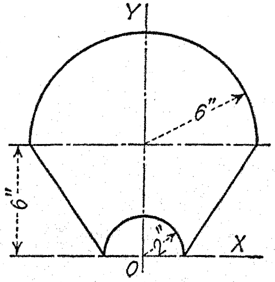


FIG. 255.

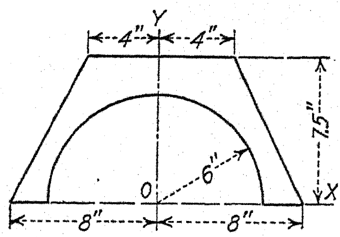


FIG. 256.

8. Solve Prob. 7 if a semicircular piece with its center midway between O and D and with a radius of 3 in. is cut out of the trapezoid.

Ans. $\bar{x} = 4.20$ in.; $\bar{y} = 2.58$ in.

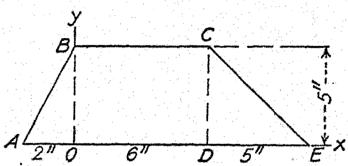


FIG. 257.

9. Solve Prob. 8 if the triangular piece AOB is bent backward about BO at right angles to the XY plane, and the triangular piece CDE is bent forward about CD at right angles to the XY plane.

Ans. $\bar{x} = 3.68$ in.; $\bar{y} = 2.58$ in.; $\bar{z} = 0.525$ in.

10. Locate the centroid of the area shown in Fig. 258.

Ans. $\bar{y} = -0.605$ ft.

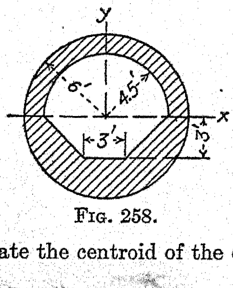


FIG. 258.

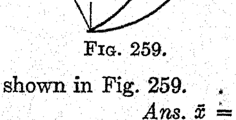


FIG. 259.

11. Locate the centroid of the crescent shown in Fig. 259.

Ans. $\bar{x} = 6.85$ in.

12. Locate the center of gravity of the governor ball and rod in the position shown in Fig. 260. The rod is steel weighing 0.284 lb./cu. in. and the sphere is cast iron weighing 0.260 lb./cu. in.

Ans. $\bar{x} = 13.22$ in.; $\bar{y} = -13.22$ in.

13. Compute the area of the surface and the volume of the solid of revolution generated by the rotation of the area ABC, Fig. 261, about the X axis.
Ans. 2007 sq. in.; 5559 cu. in.

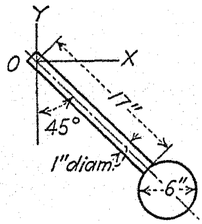


FIG. 260.

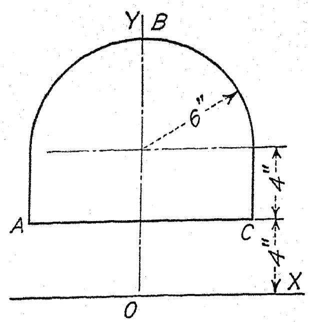


FIG. 261.

14. A flywheel 10 ft. in diameter has a rim with cross section as shown in Fig. 262. Compute the surface and the volume of the rim.
Ans. 15,736 sq. in.; 15,296 cu. in.

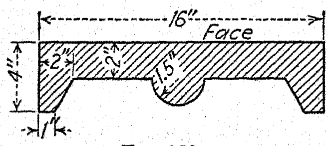


FIG. 262.

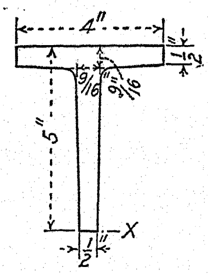


FIG. 263.

15. A steel cylinder 6 in. in diameter and 12 in. high has a concentric conical hole 5 in. in diameter and 10 in. deep cut out of the upper end and filled with lead. Locate the center of gravity from the lower base. (Steel weighs 0.284 lb./cu. in. Lead weighs 0.409 lb./cu. in.) *Ans.* 6.28 in.

16. A steel block is 8 in. square and 12 in. high. A cylindrical hole 6 in. deep is to be drilled from the upper end. Compute the necessary diameter of the hole required so that the center of gravity of the block will be 5 in. from the bottom of the block. *Ans.* 6.38 in.

17. The flange of a T beam is 8 in. wide and 1.5 in. deep, and the stem is 1.5 in. wide and 10 in. deep. Locate the centroid of the area of the cross section from the lower edge of the stem. *Ans.* 7.56 in.

18. Compute $\bar{y}$ for the standard T section shown in Fig. 263. Neglect the fillets. *Ans.* 3.44 in.

19. The T section shown in Fig. 264 is one-half of a standard 24-in. 79.9-lb. I beam. Compute $\bar{y}$, neglecting fillets and rounded corners.

Ans. 8.69 in.

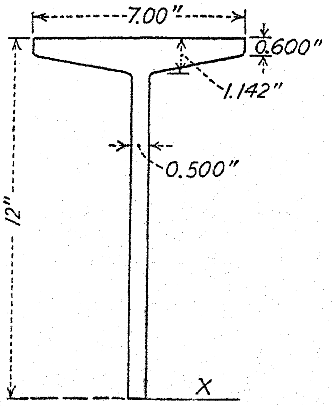


FIG. 264.

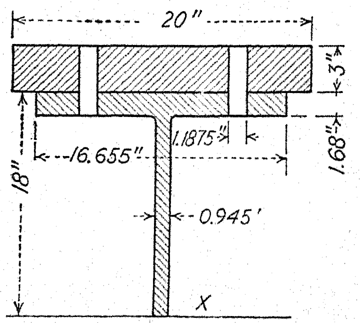


FIG. 265.

20. The T section shown in Fig. 265 consists of one-half of a 36-in. 300-lb wide flange rolled beam and a 20- by 3-in. plate, with $1\frac{3}{16}$-in. rivet holes Compute $\bar{y}$. Neglect the fillets.

Ans. 17.0 in.

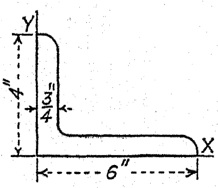


FIG. 266.

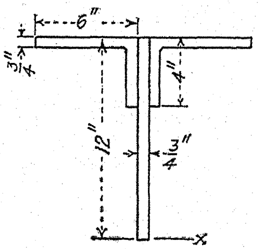


FIG. 267.

21. Locate the centroid of the standard angle section shown in Fig. 266 Neglect fillet and rounded corners.

Ans. $\bar{x} = 2.08$ in.; $\bar{y} = 1.08$ in.

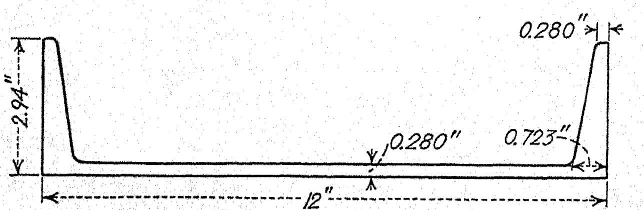


FIG. 268.

22. Locate the centroid of the built-up T section shown in Fig. 267.

Ans. 8.98 in.

23. Locate the centroid of the structural channel section shown in Fig. 268. Neglect fillets and rounded corners. *Ans.* $\bar{y} = 0.70$ in.

24. Two structural channels such as that shown in Fig. 268 are welded together as shown in Fig. 269 to form a column. Locate the centroid. *Ans.* $\bar{y} = 9.35$ in.

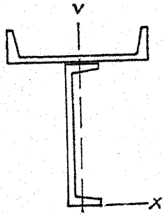


FIG. 269.

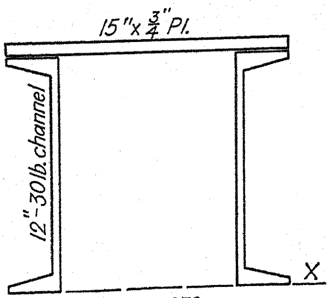


FIG. 270.

25. Figure 270 shows the cross section of the end chord of a bridge. The area of one channel is 8.79 sq. in. Locate the centroid of the cross section. *Ans.* $\bar{y} = 8.48$ in.

CHAPTER X

MOMENT OF INERTIA OF AREAS

82. Definition of Moment of Inertia and Radius of Gyration.—Integral quantities in the form $\int x^2\,dA$, $\int y^2\,dA$, and $\int \rho^2\,dA$ occur in the study of mechanics of materials. In the expression $\int x^2\,dA$, dA denotes any differential area, each part of which is the same distance x from the axis of reference, called the *inertia axis*. The sum of these differential areas equals the total area A. The quantity $\int x^2\,dA$, integrated between the proper limits, is called the *moment of inertia*[1] of the area A with respect to the Y axis.

Defined in words, the *moment of inertia* of a plane area with respect to any axis is the sum of the products of each elementary area and the square of its distance from the inertia axis.

The only axes used are those in the plane of the area and those normal to it. The moment of inertia of an area with respect to an axis in its plane, either X or Y, is called the *rectangular*

[1] The terms second moment of area, second moment of mass, etc., would be preferable, but the term moment of inertia has been in use too long to be changed. In the case of areas, the term is entirely misleading; for since an area has no inertia, it can have no true moment of inertia.

The term was first used by Euler for second moments of mass, on account of the analogy between rotary and translatory motion.

$$\frac{\text{Force}}{\text{Mass (or inertia)}} = \text{acceleration (translatory)}$$

$$\frac{\text{Moment of force}}{\int r^2\,dM \text{ (or moment of inertia)}} = \text{acceleration (rotary)}$$

According to modern definition, however, inertia is not synonymous with mass but is only a property of matter, its amount being proportional to the mass. For lack of a better name, the same term was applied to the expression $\int x^2\,dA$ for areas.

moment of inertia and is denoted by I. If it is necessary to specify the axis of reference, a subscript letter is used, as I_X, I_Y.

$$I_X = \int y^2 \, dA$$
$$I_Y = \int x^2 \, dA$$

The moment of inertia of an area with respect to an axis normal to its plane is called the polar moment of inertia and is denoted by J.

$$J = \int \rho^2 \, dA$$

An expression such as $y^2 \, dA$ or $\rho^2 \, dA$ is the product of an area and a distance squared; hence the moment of inertia of an area is expressed in a dimension of length raised to the fourth power. In numerical computations the inch is commonly used as the unit length, and moment of inertia is in units of "biquadratic inches," written in.^{4}.

It is sometimes convenient to express a moment of inertia of an area in terms of the total area and the square of a distance. Thus,

$$I_X = \int y^2 \, dA = k^2 A$$

The quantity k is called the *radius of gyration* and is the distance from the axis at which all the area could be considered as located and the moment of inertia remain the same. Stated in another way, k^2 is the mean value of x^2 for equal differential areas. As commonly determined,

$$k = \sqrt{\frac{I}{A}}$$

83. Moment of Inertia of Some Simple Figures.—In applying the expression for the moment of inertia, there are usually several possible choices of differential area dA that can be used. It is necessary that each part of the differential area selected shall be the same distance (x, y, or ρ) from the inertia axis. The differential area $dx \, dy$ may be used in rectangular coordinates but requires two integrations. Similarly, in polar coordinates, the expression $\rho \, d\theta \, d\rho$ may be used but also requires two integrations.

Larger differential areas may sometimes be used, as explained in the following examples.

EXAMPLE 1

Derive the expressions for the moment of inertia and radius of gyration of a rectangle that has base b and altitude h, with respect to its base.

Solution.—The rectangle is shown in Fig. 271. The differential area $dx\,dy$ could be used, but in this case it is simpler to use $b\,dy$. This can be done because all of the dA is the same distance y from the inertia axis X, and by so doing only one integration is required.

$$I_X = \int y^2\,dA$$

The limits of y are 0 and h, so

$$I_X = b\int_0^h y^2\,dy$$

$$I_X = \frac{b}{3}\Big[y^3\Big]_0^h$$

$$I_X = \frac{1}{3}bh^3$$

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{bh^3}{3bh}} = \frac{h}{\sqrt{3}} = 0.577h$$

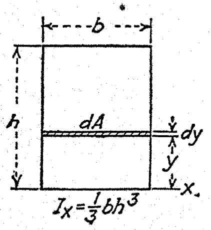


FIG. 271.

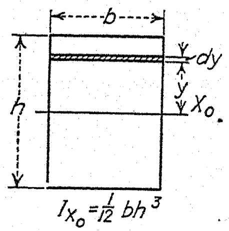


FIG. 272.

EXAMPLE 2

Derive the expressions for the moment of inertia and radius of gyration of a rectangle with respect to its centroidal axis parallel to its base.

Solution.—The rectangle is shown in Fig. 272. As in Example 1, $dA = b\,dy$. The limits of y are $-\frac{h}{2}$ and $+\frac{h}{2}$.

$$I_{X_0} = b\int_{-\frac{h}{2}}^{+\frac{h}{2}} y^2\,dy$$

$$I_{X_0} = \frac{b}{3}\Big[y^3\Big]_{-\frac{h}{2}}^{+\frac{h}{2}}$$

$$I_{X_0} = \frac{1}{12}bh^3$$

$$k_0 = \sqrt{\frac{bh^3}{12bh}} = \frac{h}{\sqrt{12}}$$

EXAMPLE 3

Derive the expressions for the moment of inertia and radius of gyration of a triangle that has base b and altitude h, with respect to its base.

Solution.—The triangle is shown in Fig. 273.

$$I_X = \int y^2\, dA$$

$$dA = u\, dy$$

By similar triangles,

$$\frac{u}{b} = \frac{h - y}{h}$$

or

$$u = b - \frac{b}{h}y$$

The limits of y are 0 and h.

$$I_X = \int_0^h by^2\, dy - \int_0^h \frac{b}{h}y^3\, dy$$

$$I_X = \frac{bh^3}{3} - \frac{bh^3}{4} = \frac{1}{12}bh^3$$

$$k = \sqrt{\frac{bh^3}{6bh}} = \frac{h}{\sqrt{6}}$$

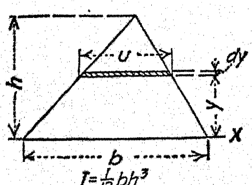


FIG. 273.

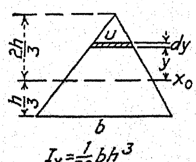


FIG. 274.

EXAMPLE 4

Derive the expressions for the moment of inertia and radius of gyration of a triangle that has base b and altitude h, with respect to its centroidal axis parallel to the base.

Solution.—The triangle is shown in Fig. 274.

$$I_{X_0} = \int y^2\, dA$$

$$dA = u\, dy$$

By similar triangles,

$$\frac{u}{b} = \frac{\tfrac{2}{3}h - y}{h}$$

$$u = \frac{2}{3}b - \frac{b}{h}y$$

The limits of y are $-\tfrac{1}{3}h$ and $+\tfrac{2}{3}h$.

$$I_{X_0} = \int_{-\frac{1}{3}h}^{+\frac{2}{3}h} y^2\left(\frac{2}{3}b - \frac{b}{h}y\right)dy$$

$$I_{X_0} = \frac{2}{3}b\int_{-\frac{1}{3}h}^{+\frac{2}{3}h} y^2\,dy - \frac{b}{h}\int_{-\frac{1}{3}h}^{+\frac{2}{3}h} y^3\,dy$$

$$I_{X_0} = \frac{2}{9}b\Big[y^3\Big]_{-\frac{1}{3}h}^{+\frac{2}{3}h} - \frac{b}{4h}\Big[y^4\Big]_{-\frac{1}{3}h}^{+\frac{2}{3}h}$$

$$I_{X_0} = \frac{2}{9}b\left(\frac{8}{27}h^3 + \frac{1}{27}h^3\right) - \frac{b}{4h}\left(\frac{16}{81}h^4 - \frac{1}{81}h^4\right)$$

$$I_{X_0} = \frac{1}{36}bh^3$$

$$k_0 = \sqrt{\frac{bh^3}{18bh}} = \frac{h}{\sqrt{18}}$$

EXAMPLE 5

Derive the expressions for the polar moment of inertia and radius of gyration of a circle of radius r with respect to an axis through its center.

Solution.

$$J_0 = \int \rho^2\,dA$$

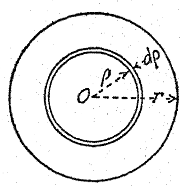


FIG. 275.

The differential area will be that shown in Fig. 275. This can be used because each part of it is the same distance ρ from the inertia axis. The limits of ρ are 0 and r.

$$dA = 2\pi\rho\,d\rho$$

$$J_0 = 2\pi\int_0^r \rho^3\,d\rho$$

$$J_0 = 2\pi\left[\frac{\rho^4}{4}\right]_0^r$$

$$J_0 = \frac{1}{2}\pi r^4$$

$$k_0 = \sqrt{\frac{\pi r^4}{2\pi r^2}} = \frac{r}{\sqrt{2}}$$

EXAMPLE 6

Derive the expressions for the moment of inertia and radius of gyration of a circle of radius r with respect to a diameter.

Solution.—Let the differential area be taken as shown in Fig. 276.

$$dA = x\,dy$$

For the quadrant,

$$I_{X_0} = \int_0^r y^2 x\,dy$$

$$x = (r^2 - y^2)^{1/2}$$

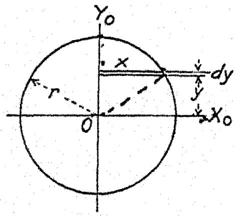


FIG. 276.

For the entire circle,

$$I_{X_0} = 4\int_0^r (r^2 - y^2)^{1/2} y^2\,dy$$

$$I_{X_0} = 4\left[\frac{y}{8}(2y^2 - r^2)(r^2 - y^2)^{1/2} + \frac{r^4}{8}\sin^{-1}\frac{y}{r}\right]_0^r$$

$$I_{X_0} = \frac{1}{4}\pi r^4$$

$$k = \sqrt{\frac{\pi r^4}{4\pi r^2}} = \frac{r}{2}$$

Problems

1. Derive the expressions for the moment of inertia and radius of gyration of a triangle with respect to an axis through the vertex parallel to the base. *Ans.* $I = \frac{1}{4}bh^3$; $k = h/\sqrt{2}$.

2. Using the differential area as $\rho\, d\theta\, d\rho$, derive the expression for the moment of inertia of a circle with respect to a diameter.

$$\left(\int \sin^2\theta\, d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)$$

3. Derive the expressions for the moment of inertia and radius of gyration of an ellipse with respect to the longer axis. The equation of the ellipse is $b^2x^2 + a^2y^2 = a^2b^2$. *Ans.* $\frac{1}{4}\pi ab^3$; $k = b/2$.

84. The Transfer Formula for Parallel Axes.—In some cases, the moment of inertia of an area with respect to a given axis in its plane may be obtained more easily by the transfer formula than by direct integration if the value of its moment of inertia with respect to any parallel axis is known. In Fig. 277, let X_0 be a centroidal axis, and X_1 any other axis parallel to it, in the plane of the area, at a distance d from the centroidal axis.

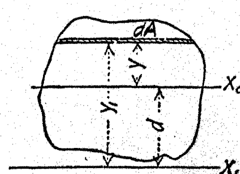


FIG. 277.

$$I_{X_1} = \int y_1^2\,dA$$

$$y_1^2 = (y + d)^2 = y^2 + 2yd + d^2$$

$$I_{X_1} = \int y^2\,dA + 2d\int y\,dA + d^2\int dA$$

$$\mathbf{I}_{X_1} = \mathbf{I}_{X_0} + \mathbf{A}\mathbf{d}^2$$

The quantity $2d\int y\,dA = 0$ because $\int y\,dA = \bar{y}A$, and for the centroidal axis $\bar{y} = 0$.

Stated in words, the equation above is as follows: The moment of inertia of an area with respect to any axis in its plane is equal to its moment of inertia with respect to a parallel centroidal axis plus the product of the area and the square of the distance between the axes.

It should be noted that the transfer formula gives the relation between the moments of inertia of an area with respect to its *centroidal* axis and any other parallel axis.

If both sides of the equation $I_{X_1} = I_{X_0} + Ad^2$ are divided by A, it becomes

$$\frac{I_{X_1}}{A} = \frac{I_{X_0}}{A} + d^2$$

Hence,

$$k_X{}^2 = k_{X_0}{}^2 + d^2$$

or

$$\mathbf{k^2 = k_0{}^2 + d^2}$$

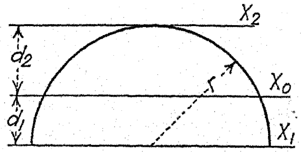


FIG. 278.

EXAMPLE

Derive the expression for the moment of inertia of a semicircle with respect to a tangent parallel to the bounding diameter.

Solution.—The semicircle is shown in Fig. 278, X_2 being the axis for which the moment of inertia is required. I_{X_1} of the semicircle is one-half the moment of inertia of a circle with respect to its diameter.

$$I_{X_1} = \frac{1}{8}\pi r^4$$

By the transfer formula,

$$I_{X_1} = I_{X_0} + Ad_1{}^2$$

$$\frac{1}{8}\pi r^4 = I_{X_0} + \frac{1}{2}\pi r^2\left(\frac{4r}{3\pi}\right)^2$$

$$I_{X_0} = \frac{1}{8}\pi r^4 - \frac{8}{9\pi}r^4$$

$$I_{X_2} = I_{X_0} + Ad_2{}^2$$

$$I_{X_2} = \frac{\pi r^4}{8} - \frac{8r^4}{9\pi} + \frac{\pi r^2}{2}\left(r - \frac{4r}{3\pi}\right)^2$$

$$I_{X_2} = \frac{5\pi r^4}{8} - \frac{4r^4}{3}$$

Problems

1. Derive the expression for the moment of inertia of a circle with respect to a tangent. *Ans.* $\frac{5}{4}\pi r^4$.

2. Given $I = \frac{1}{12}bh^3$ for a triangle with respect to its base, derive the expression for the moment of inertia with respect to the axis through the vertex parallel to the base.

3. Derive the expression for I_X of the semicircle shown in Fig. 279

Ans. $\frac{5\pi r^4}{8} + \frac{4r^4}{3}$.

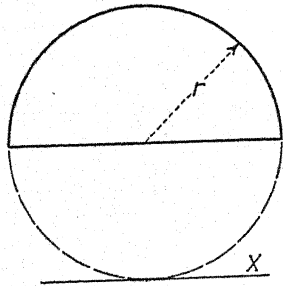


FIG. 279.

85. Polar Moment of Inertia.—Let Fig. 280 represent any plane area. With respect to the polar axis through O,

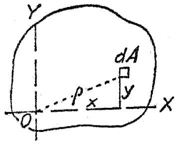


FIG. 280.

$$J = \int \rho^2\, dA$$
$$\rho^2 = x^2 + y^2$$
$$J = \int (x^2 + y^2) dA$$
$$J = \int x^2\, dA + \int y^2\, dA$$

The quantity $\int x^2\, dA = I_Y$ and $\int y^2\, dA = I_X$

$$J = I_Y + I_X$$

The polar moment of inertia of an area with respect to any axis equals the sum of its moments of inertia with respect to any two rectangular axes in the area intersecting the polar axis.

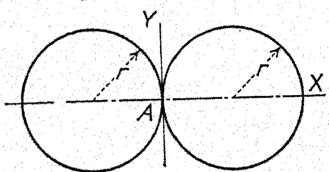


FIG. 281.

EXAMPLE

Compute the polar moment of inertia of the two circles shown in Fig. 281 with respect to point A, their common point of tangency.

Solution.—For the two areas,

$$I_X = \frac{1}{2}\pi r^4 \text{ and } I_Y = \frac{5}{2}\pi r^4$$
$$J = I_X + I_Y = 3\pi r^4$$

Problems

1. Derive the expression for the polar moment of inertia of a rectangle with base b and height h with respect to its centroid.

Ans. $J = \frac{bh}{12}(b^2 + h^2)$.

2. Prove that the moment of inertia of a square with respect to any centroidal axis in its plane is $b^4/12$.

3. Given $J = \frac{1}{2}\pi r^4$ for a circle with respect to an axis through the center, show that $I_X = \frac{1}{4}\pi r^4$ with respect to a diameter.

86. The Transfer Formula for Parallel Polar Axes.—The relation between the polar moment of inertia of an area with respect to a centroidal axis and that with respect to any parallel axis is similar to that between moments of inertia with respect to parallel axes in the plane of the area.

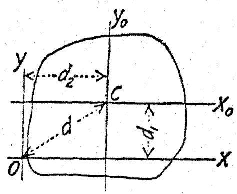


FIG. 282.

Let X_0 and Y_0, Fig. 282, be the centroidal axes, and X and Y any other parallel axes, all of them being in the plane of the area. By Art. 84,

$$I_X = I_{X_0} + Ad_1^2$$

and

$$I_Y = I_{Y_0} + Ad_2^2$$

Let Z be the axis through O perpendicular to X and Y, and let Z_0 be the axis through C perpendicular to X_0 and Y_0. By Art. 85,

$$J_0 = I_X + I_Y = I_{X_0} + I_{Y_0} + A\ (d_1^2 + d_2^2)$$

Since

$$I_{X_0} + I_{Y_0} = J_C, \text{ and } d_1^2 + d_2^2 = d^2$$
$$\mathbf{J_0 = J_C + Ad^2}$$

The polar moment of inertia of an area with respect to any axis is equal to its polar moment of inertia with respect to the centroidal axis plus the product of the area and the square of the distance between the two axes.

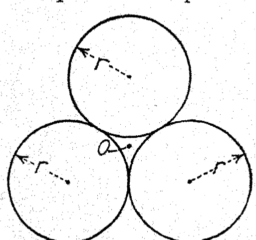


FIG. 283.

Problems

1. Compute the polar moment of inertia of the three circles shown in Fig. 283 with respect to point O, the point equidistant from the three centers. *Ans.* $5.5\pi r^4$.

2. Compute the polar moment of inertia of the three circles shown in Fig. 283 with respect to the center of any one of the three circles. *Ans.* $9.5\pi r^4$.

87. Computation of Moment of Inertia of Simple Areas.—Since the squares of both positive and negative quantities are

positive, and since areas are always positive, it follows that moments of inertia are always positive. It follows also that the moment of inertia of an area on one side of an axis is the same as it would be if it were symmetrically placed on the other side of the axis. Thus the moment of inertia of each half of a circle with respect to any diameter is the same.

In computing the moment of inertia of an area, the expressions derived in the preceding articles are used as needed. If there is a choice of method, that one should be used which gives the least amount of computation.

Problems

1. A rectangle is 16 in. wide and 6 in. deep. Compute its moment of inertia (1) with respect to its centroidal axis parallel to its base; (2) with respect to its base; and (3) with respect to an axis parallel to the base and 4 in. below it. *Ans.* (1) 288 in.^{4}; (2) 1152 in.^{4}; (3) 4992 in.^{4}.

2. Compute the moment of inertia and radius of gyration of a circle 1.2 in. in diameter with respect to a diameter. *Ans.* 0.10179 in.^{4}; 0.3 in.

3. Compute the polar moment of inertia of a circle 3 in. in diameter with respect to its center. *Ans.* 7.955 in.^{4}.

4. A rectangle has a base b and height h. Show that the moment of inertia with respect to the centroidal axis parallel to base b is to that with respect to the centroidal axis parallel to height h as h^2 is to b^2.

5. Compute the polar moment of inertia of an equilateral triangle 3 in. on a side with respect to its centroid. *Ans.* 2.923 in.^{4}.

88. Computation of Moment of Inertia of Composite Areas.—The moment of inertia of a composite area with respect to any axis equals the sum of the moments of inertia of the separate parts with respect to the same axis. For example, the moment of inertia of the trapezoid $ABCD$, Fig. 284, with respect to the base AD is the sum of the moments of inertia of the rectangle $FBCE$ and the two triangles ABF and ECD with respect to AD. The moment of inertia with respect to axis AD having been computed, that with respect to the centroidal axis X_0 can be obtained by the transfer formula; then that with respect to any other parallel axis.

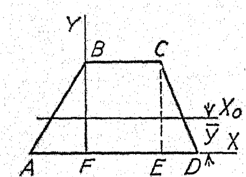


FIG. 284.

If an area is obtained by removing a simple part from an area that was originally simple, the moment of inertia of the part remaining with respect to any given axis is equal to the moment

of inertia of the original area minus the moment of inertia of the part removed. The moment of inertia of the annulus (Fig. 285) with respect to a diameter is equal to the moment of inertia of the larger circle minus the moment of inertia of the smaller circle with respect to the same axis.

$$I_X = \tfrac{1}{4}\pi r_2^4 - \tfrac{1}{4}\pi r_1^4$$

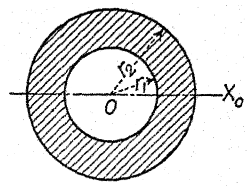


FIG. 285.

Problems

1. In Fig. 284, let $BC = 8$ in., $BF = 10$ in., $AF = 6$ in., and $ED = 4$ in. Compute the moment of inertia of the trapezoid with respect to its base AD. Compute $\bar{y}$. Compute I_{X_c} by the transfer formula.

Ans. $I_X = 3500$ in.^{4}; $\bar{y} = 4.359$ in.; $I_{X_0} = 1029$ in.^{4}.

2. Solve for the moment of inertia of the trapezoid (Fig. 284) by getting it first with respect to axis BC, then transferring to axis X_0.

3. Solve for the moment of inertia of the trapezoid (Fig. 284) a third way by getting first the moment of inertia of each component part with respect to its own centroidal axis parallel to the axis X_0, then transferring to axis X_0 and adding.

4. Compute I_{X_0} of the annulus shown in Fig. 285 if $r_1 = 5$ in. and $r_2 = 9$ in.

Ans. 4662 in.^{4}.

5. Locate the centroid of the upper half of the annulus shown in Fig. 285. Compute its moment of inertia with respect to its centroidal axis parallel to its diameter. *Ans.* 4.573 in.; 491 in.^{4}.

6. In Fig. 285, assume only the lower half of the inner circle to be cut out, the upper half being solid. Locate the new centroidal axis and compute the moment of inertia with respect to it.

Ans. $\bar{y} = 0.387$ in.; $I_{X_0} = 4876$ in.^{4}.

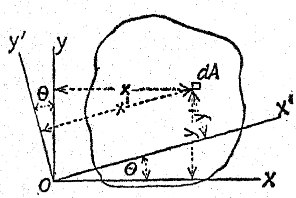


FIG. 286.

89. Moment of Inertia with Respect to Inclined Axes.—In Fig. 286, let X and Y be any two rectangular axes for which $I_X = \int y^2\,dA$ and $I_Y = \int x^2\,dA$. X' and Y' are axes at the angle θ with the original pair. Then

$$I_X' = \int (y')^2\,dA \quad \text{and} \quad I_Y' = \int (x')^2\,dA$$

Also $y' = y\cos\theta - x\sin\theta$, and $x' = x\cos\theta + y\sin\theta$, from the geometry of the figure. By squaring these values and substituting above,

$$I_X' = \int y^2 \cos^2\theta\,dA - 2\int xy\cos\theta\sin\theta\,dA + \int x^2\sin^2\theta\,dA$$

By integration,

$$I_X' = \cos^2\theta \cdot I_X + \sin^2\theta \cdot I_Y - 2\cos\theta\sin\theta\int xy\,dA$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2},\ \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

and

$$2\cos\theta\sin\theta = \sin 2\theta$$

By substitution of these values the equation above becomes

$$I_X' = \frac{I_X + I_Y}{2} + \frac{I_X - I_Y}{2}\cos 2\theta - \sin 2\theta\int xy\,dA \qquad (1)$$

Similarly,

$$I_Y' = \frac{I_X + I_Y}{2} - \frac{I_X - I_Y}{2}\cos 2\theta + \sin 2\theta\int xy\,dA \qquad (2)$$

By adding these expressions for I_X' and I_Y', it is found that

$$I_X' + I_Y' = I_X + I_Y$$

as shown in Art. 85.

Equations (1) and (2) simplify what would otherwise be a very tedious operation. If the moments of inertia of an area with respect to any two rectangular axes in the plane of the area are known, the moment of inertia with respect to any coplanar inclined axis passing through their point of intersection may be easily computed.

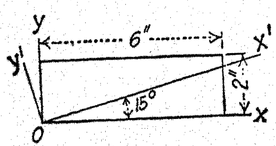


FIG. 287.

EXAMPLE

Compute the moment of inertia of a rectangle 6 in. wide and 2 in. high with respect to an axis through the lower left-hand corner at an angle of 15° with the base, as shown in Fig. 287. Compute also I_Y'.

Solution.

$$I_X = 16;\ I_Y = 144$$

$$\int xy\,dA = \int_0^6\int_0^2 x\,dx\,y\,dy = 36$$

$$I_X' = 80 - 64\cos 30° - 36\sin 30°$$

$$I_X' = 6.58\text{ in.}^4$$

$$I_Y' = 80 + 64\cos 30° + 36\sin 30°$$

$$I_Y' = 153.42\text{ in.}^4$$

Problems

1. Compute the moment of inertia of a 6-in. square with respect to its diagonal by the method of this article. *Ans.* 108 in.^{4}.

2. Compute the moment of inertia of a rectangle 8 in. wide and 6 in. high with respect to its diagonal. *Ans.* 184.32 in.^{4}.

3. Compute the moment of inertia of the rectangle shown in Fig. 287 with respect to an axis through O at an angle of $-7°30'$ with the X axis. *Ans.* 27.49 in.^{4}.

90. Product of Inertia.—By analogy with moment of inertia, the expression $\int xy\,dA$ is called the *product of inertia* of the area and is denoted by H. The form of the expression shows that product of inertia is always taken with respect to a *pair* of rectangular axes.

If either one of the axes is an axis of symmetry for the area, the product of inertia with respect to that pair of axes is zero.

Proof.—Let Fig. 288 be any area symmetrical with respect to the Y axis.

$$H = \int xy\,dA$$

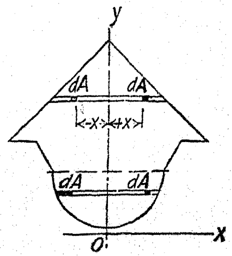


FIG. 288.

In the summation of the products $xy\,dA$, it will be seen that for each term $(+x)y\,dA$ there is a numerically equal term $(-x)y\,dA$ to neutralize it. Hence, for a figure symmetrical with respect to the Y axis,

$$H = \int xy\,dA = 0$$

Similarly, $H = 0$ for a figure symmetrical with respect to the X axis.

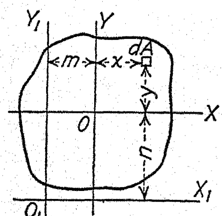


FIG. 289.

91. The Transfer Formula for Products of Inertia.—After the product of inertia is determined with respect to a pair of rectangular centroidal axes, it may be calculated easily with respect to any other pair of parallel axes.

In Fig. 289, OX and OY are any two rectangular centroidal axes; O_1X_1 and O_1Y_1 are any other pair of parallel axes in the same plane; (x, y) are the coordinates of dA with respect to the original axes; and $(m + x, n + y)$ are the coordinates of dA with respect to the new axes.

$$H_{O_1} = \int (m + x)(n + y)dA$$

$$H_{0_1} = \int mn\,dA + \int my\,dA + \int nx\,dA + \int xy\,dA$$
$$H_{0_1} = mnA + 0 + 0 + H_0$$

H_0 is the product of inertia of the area with respect to the original axes.

This expression is similar to the transfer formula for moment of inertia, d^2 being replaced by mn.

The quantities m and n may be either positive or negative, so the term mnA may be either positive or negative. If the centroid of the area is in the first or third quadrant of the axes with respect to which H is taken, mnA is positive; if in the second or fourth quadrant, it is negative.

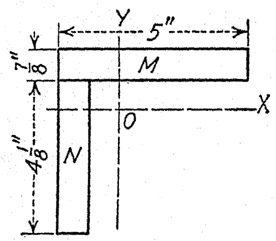


FIG. 290.

As in the case of moment of inertia, the product of inertia of an area composed of several simple parts with respect to any pair of axes is equal to the algebraic sum of the products of inertia of the several parts with respect to the same axes. For example, if H_{XY} of the angle section of Fig. 290 is required, the area may be divided into the two rectangles M and N. Then H_{XY} of the angle section $= H_{XY}$ of $M + H_{XY}$ of N.

EXAMPLE

Determine the value of H_{0_1} for the right triangle shown in Fig. 291.

Solution.

$$\begin{aligned} H_0 &= \int xy\,dA \\ &= \int_{-4}^{+2}\int_{-1}^{\frac{x}{2}+1} x\,dxy\,dy \\ &= 4.5 \text{ in.}^4 \\ H_{0_1} &= H_0 + mnA \\ &= +4.5 + 36 \\ &= +40.5 \text{ in.}^4 \end{aligned}$$

In this case, H_{0_1} can be determined more easily by integrating directly than by the transfer method used above.

$$\begin{aligned} H_{0_1} &= \int_0^6\int_0^{\frac{x}{2}} x\,dxy\,dy \\ &= +40.5 \text{ in.}^4 \end{aligned}$$

If $H_0 = 0$, as it does if either X or Y is an axis of symmetry, the transfer method is much simpler, for then

$$H_{0_1} = mnA$$

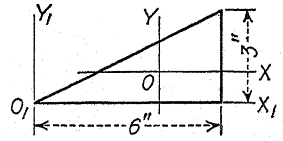


FIG. 291.

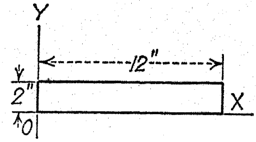


FIG. 292.

Problems

1. Compute the value of H_0 for the rectangle shown in Fig. 292. *Ans.* $+144$ in.^{4}.

2. In Fig. 290, the distance from the back of the angle to the centroidal axis is 1.57 in. Compute H_0. *Ans.* $+10.20$ in.^{4}.

3. Compute the product of inertia of an 8- by 6- by 1-in. angle section with respect to the centroidal axes parallel to the legs. *Ans.* ± 32.31 in.^{4}.

92. Maximum and Minimum Moments of Inertia.—The moment of inertia of an area with respect to an axis at an angle θ with some original axis is given by equation (1), Art. 89.

$$I_X' = \frac{I_X + I_Y}{2} + \frac{I_X - I_Y}{2} \cos 2\theta - H \sin 2\theta$$

As θ varies, the value of I_X' varies. The values of θ for maximum and minimum values of I_X' are determined by differentiating the expression for I_X' and placing the first derivative equal to zero.

$$\frac{dI_X'}{d\theta} = (I_Y - I_X) \sin 2\theta - 2H \cos 2\theta$$

For the maximum or minimum value of I_X', $dI_X'/d\theta = 0$. Then

$$\tan 2\theta = \frac{2H}{I_Y - I_X}$$

Two values of 2θ differing by 180° are obtained from the equation above, and therefore two values of θ differing by 90°. One value gives the angle for maximum I_X'; the other, the value for minimum I_X'. The maximum and minimum moments of inertia are called the *principal moments of inertia;* and the corresponding axes, the *principal axes.*

If either the X or the Y axis is an axis of symmetry, $H = 0$, by Art. 90; therefore $\tan 2\theta = 0$. $2\theta = 0°$ or 180°, and $\theta = 0°$ or 90°, so the X and Y axes are the principal axes.

EXAMPLE

Compute the maximum and minimum moments of inertia of a rectangle 6 in. wide and 4 in. high with respect to axes through the lower left-hand corner, as shown in Fig. 293.

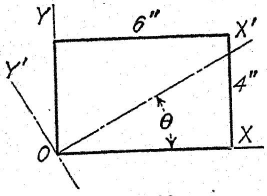


FIG. 293.

Solution.

$$I_Y = \tfrac{1}{3} \times 4 \times 6^3 = 288 \text{ in.}^4$$
$$I_X = \tfrac{1}{3} \times 6 \times 4^3 = 128 \text{ in.}^4$$
$$H_0 = mnA = 3 \times 2 \times 24 = 144 \text{ in.}^4$$
$$\tan 2\theta = \frac{2 \times 144}{288 - 128} = 1.8$$
$$2\theta = 60°57', \text{ or } 240°57'$$
$$\theta = 30°28', \text{ or } 120°28'$$

For $\theta = 30°28'$, the moment of inertia is a minimum.
For $\theta = 120°28'$, the moment of inertia is a maximum.
For $\theta = 30°28'$, $\cos 2\theta = 0.4857$, and $\sin 2\theta = 0.8742$.

$$\text{Min. } I_X' = \frac{128 + 288}{2} + \frac{128 - 288}{2} \times 0.4857 - 144 \times 0.8742 = 43.26 \text{ in.}^4.$$

For $\theta = 120°28'$, $\cos 2\theta = -0.4857$, and $\sin 2\theta = -0.8742$.

$$\text{Max. } I_X' \ (I_Y') = \frac{128 + 288}{2} - \frac{128 - 288}{2} \times 0.4857 + 144 \times 0.8742 = 372.74 \text{ in.}^4.$$

Problems

1. Compute the maximum and minimum moments of inertia of the rectangle shown in Fig. 292 with respect to axes through O.
Ans. 1170.22 in.^{4}; 13.78 in.^{4} ($\theta = 7°13'$ or $97°13'$).

2. Compute the maximum and minimum moments of inertia of the angle section shown in Fig. 290 with respect to centroidal axes.
Ans. 28.0 in.^{4}; 7.6 in.^{4} ($\theta = 45°$ or $135°$).

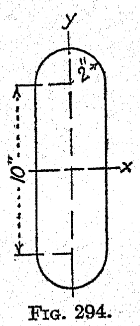


FIG. 294.

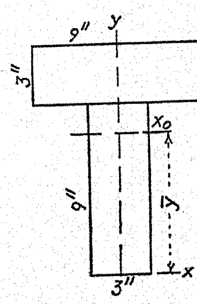


FIG. 295.

3. Compute the maximum and minimum moments of inertia of an 8- by 6- by 1-in. angle section with respect to centroidal axes.
Ans. 98.32 in.^{4}; 21.28 in.^{4} ($\theta = 28°30'$ or $118°30'$).

GENERAL PROBLEMS ON MOMENT OF INERTIA OF AREAS

1. Compute the moments of inertia of the area shown in Fig. 294 with respect to the X and Y axes. *Ans.* 767 in.^{4}; 65.9 in.^{4}.

2. Locate the centroidal axis X_0 of the area shown in Fig. 295. Compute the moments of inertia with respect to axes X, X_0, and Y.

Ans. 7.5 in.; 3726 in.^{4}; 688.5 in.^{4}; 202.5 in.^{4}.

3. Compute the moment of inertia and radius of gyration of the area shown in Fig. 296. Show that these values are true for any centroidal axes in the plane of the area. *Ans.* 484.15 in.^{4}; 2.64 in.

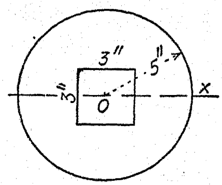


FIG. 296.

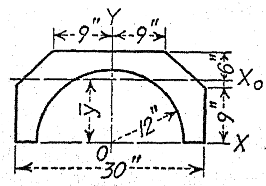


FIG. 297.

4. Compute the moments of inertia of the area shown in Fig. 297 with respect to the X and Y axes. *Ans.* I_X = 19,450 in.^{4}; I_Y = 19,450 in.^{4}.

5. Compute $\bar{y}$ and I_{X_0} for the area shown in Fig. 297.

Ans. $\bar{y}$ = 9.35 in.; I_{X_0} = 3040 in.^{4}.

6. Derive the expression for I_Y of the circular sector shown in Fig. 298.

Ans. $I_Y = \dfrac{r^4}{8}(\alpha + \sin\alpha)$.

7. Derive the expression for I_X of the circular sector shown in Fig. 298.

Ans. $I_X = \dfrac{r^4}{8}(\alpha - \sin\alpha)$.

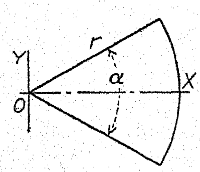


FIG. 298.

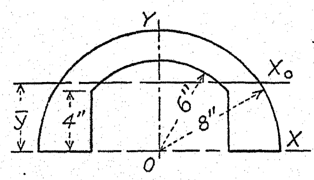


FIG. 299.

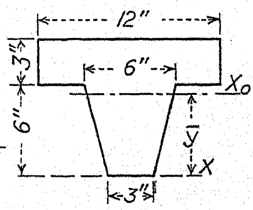


FIG. 300.

8. Compute $\bar{y}$, I_X, and I_{X_0} of the area shown in Fig. 299.

Ans. $\bar{y}$ = 4.01 in.; I_X = 1128 in.^{4}; I_{X_0} = 286 in.^{4}.

9. Compute I_Y of the area shown in Fig. 299. *Ans.* 1319 in.^{4}.

10. Compute $\bar{y}$ and I_{X_0} for the area shown in Fig. 300.

Ans. $\bar{y}$ = 5.714 in.; I_{X_0} = 372.86 in.^{4}.

11. Compute $\bar{y}$ and I_{X_0} for the section shown in Fig. 301.

Ans. $\bar{y}$ = 4.75 in.; I_{X_0} = 330.53 in.^{4}.

12. The dimensions of the standard wide-flange column core section are shown in Fig. 302. Compute I_X and I_Y.

Ans. $I_X = 4142$ in.^{4}; $I_Y = 1635$ in.^{4}.

NOTE.—Neglect fillets and rounded corners in the cross sections of structural shapes.

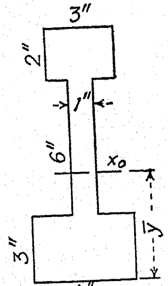


FIG. 301.

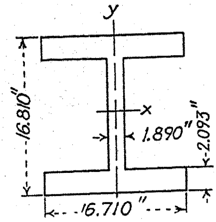


FIG. 302.

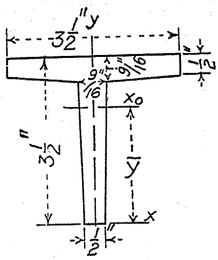


FIG. 303.

13. Compute $\bar{y}$ and I_{X_0} of the T section shown in Fig. 303.

Ans. $\bar{y} = 2.45$ in.; $I_{X_0} = 3.73$ in.^{4}.

14. Locate the positions of the gravity axes parallel to the legs of a 6- by 4- by ⅝-in. angle section, and compute the moment of inertia of the cross section with respect to each axis.

Ans. 2.03 in.; 1.03 in.; 21.1 in.^{4}; 7.5 in.^{4}.

15. The section shown in Fig. 304 is composed of four 6- by 4- by ⅝-in. angles and a web plate 12 by ⅝ in. The 4-in. legs are fastened to the web plate and height $h = 12\frac{1}{2}$ in. Compute I_X, I_Y, k_X, and k_Y.

Ans. 758.7 in.^{4}; 213.2 in.^{4}; 4.95 in.; 2.63 in.

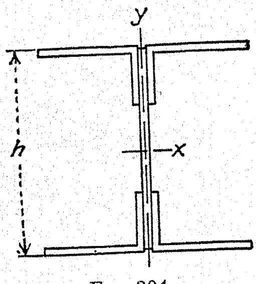


FIG. 304.

16. Solve Prob. 15 if two 14- by 2-in. flange plates are added.

Ans. 3721 in.^{4}; 1128 in.^{4}; 6.54 in.; 3.60 in.

17. Compute $\bar{y}$, I_{X_0}, and I_Y of the 10-in. 15.3-lb. channel section shown in Fig. 305.

Ans. 0.64 in.; 2.31 in.^{4}; 66.87 in.^{4}.

18. Each of the three channels shown in Fig. 306 is the same as the one shown in Fig. 305. Using the values obtained in Prob. 17, compute the values of I_X and I_Y. The area of one channel is 4.47 sq. in.

Ans. 355.9 in.^{4}; 136.1 in.^{4}.

19. In Fig. 307, each channel is the same as the one shown in Fig. 305, and the flange plates are ½ in. thick. Compute the width of the plates necessary so that $I_X = I_Y$. Compute I_X.

Ans. 14.84 in.; 543 in.⁴.

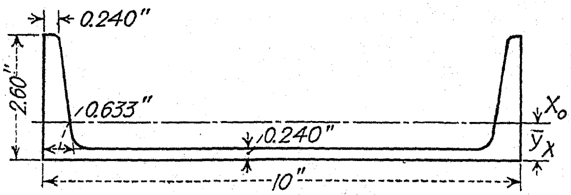


FIG. 305.

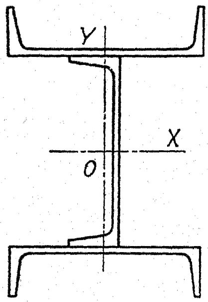


FIG. 306.

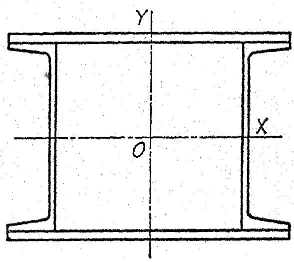


FIG. 307.

20. Compute I_X and I_Y of the symmetrical Z-bar section shown in Fig. 308.

Ans. 9.1 in.⁴; 19.2 in.⁴.

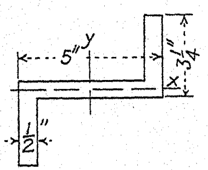


FIG. 308.

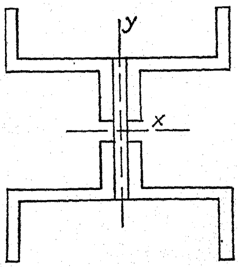


FIG. 309.

21. The section shown in Fig. 309 consists of four 5- by 3¼- by ½-in. Z bars and a 7- by ½-in. web plate. Compute I_X and I_Y. See Prob. 20 for data on one Z bar. Area of one Z bar = 5.25 sq. in.

Ans. 272.5 in.⁴; 235.7 in.⁴.

22. Compute I_X and I_Y of the I-beam section shown in Fig. 310.

Ans. 268.9 in.⁴; 13.8 in.⁴.

23. The section shown in Fig. 311 consists of two 8- by 8- by ⅝-in. angles riveted to the web of a 12-in. 40.8-lb. I beam. Locate the centroid, and

compute the moment of inertia of the section with respect to the X_0 axis. See Prob. 22 for I_X of I beam. *Ans.* 7.1 in.; 410.6 in.^{4}.

24. Consider that an 8- by 1-in. flange plate is riveted to the top flange of the I beam shown in Fig. 310. Compute $\bar{y}$ from the bottom and the value of I_{X_0}. *Ans.* 8.62 in.; 471.3 in.^{4}.

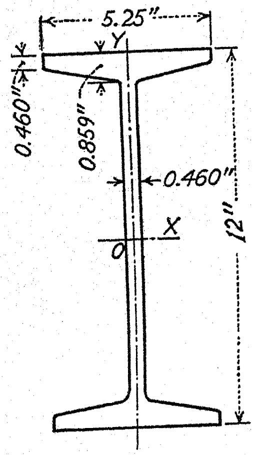


FIG. 310.

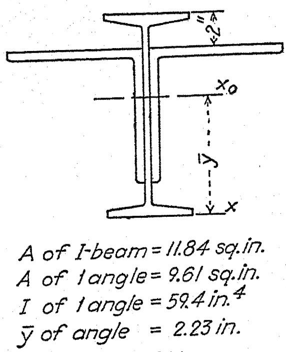


FIG. 311.

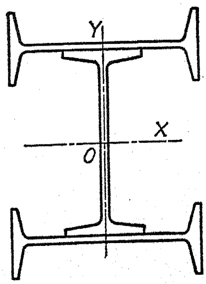


FIG. 312.

25. The section shown in Fig. 312 consists of three 12-in. 40.8-lb. I beams riveted together. Compute I_X and I_Y. *Ans.* 1215.6 in.^{4}; 551.6 in.^{4}.

26. Prove that the moment of inertia of any equal-sided area (equilateral triangle, square, pentagon, hexagon, etc.) with respect to any centroidal axis in its plane is constant.

PART II. DYNAMICS

CHAPTER XI

KINEMATICS OF A PARTICLE

93. Definitions.—*Dynamics* is defined as that part of Mechanics which treats of bodies in motion. *Kinematics* is that part of dynamics which treats of the motion of bodies without reference to the forces causing or changing the motion. *Kinetics* is that part of dynamics which treats of the effects of forces in causing the motion or in changing the motion of bodies.

A *particle* is defined as a body so small that the matter it contains may be considered to have position only, as a geometric point. If the size of a material body is small compared to its range of motion, it may be considered as a particle.

94. Kinds of Motion.—The two kinds of motion most commonly considered are *rectilinear* motion and *curvilinear* motion. As the term implies, rectilinear motion of a particle is motion along a straight line. Curvilinear motion is motion along a curved path. In general, the treatment in this book will be of *plane* curvilinear motion only.

95. Rectilinear Displacement.—If a particle has rectilinear motion with respect to some point which is assumed to be fixed, its *displacement* is its total change of position during any given interval of time. The point of reference usually assumed is one which is at rest with respect to the surface of the earth.

Displacement is independent of the path traversed and depends only upon the initial and final positions of the particle. Since displacement necessarily has direction as well as magnitude, it is a vector quantity and is represented graphically by a vector.

The unit of displacement most commonly used in engineering work is the foot, although any unit of length may be used.

The unit of time most commonly used in engineering work is the second, although any unit of time may be used.

Problems

1. The equation of motion of a particle is $s = 16.1t^2$, displacement s being in feet, and time t being in seconds. Compute the displacement of

the particle after 3 sec. Compute the displacement during the third second. Compute the displacement during the latter half of the third second. *Ans.* 144.9 ft.; 80.5 ft.; 44.28 ft.

2. The equation of motion of a particle is $s = -10 + 3t + 8t^2$, s being in feet and t in seconds. Compute the time when the displacement is zero and the time when the displacement is 10 ft. *Ans.* 0.946 sec.; 1.405 sec.

96. Rectilinear Velocity and Speed.—The velocity of a particle with rectilinear motion is the time rate of its displacement from some assumed point of reference. If the particle traverses equal spaces in equal time intervals, its velocity is uniform, and the amount of its velocity is equal to the ratio of the given displacement to the time in which the displacement was made. If s is the displacement and t the time, the amount of the velocity v is given by the equation

$$v = \frac{s}{t}$$

If the unit of displacement is the foot, and the unit of time is the second, the unit of velocity will be the *foot per second*, abbreviated ft./sec. If the unit of displacement is the mile, and the unit of time is the hour, the unit of velocity will be the *mile per hour*, abbreviated m.p.h.

If a particle moves over unequal spaces in equal time intervals, its velocity is variable. In this case the ratio of any given space s to the time t in which it was traversed gives only the *average* velocity. As the space s is shortened until it becomes ds, and the time in which ds is traversed becomes dt, this average velocity approaches the value of the instantaneous velocity at the point where ds is taken. This instantaneous velocity is

$$\mathbf{v} = \frac{\mathbf{ds}}{\mathbf{dt}}$$

Velocity has direction as well as magnitude. It is therefore a *vector* quantity and is represented graphically by a vector. *Speed* is the *scalar*, or *quantity*, part of velocity and is merely the rate of travel, irrespective of direction. The units of speed are the same as the units of velocity.

Problems

1. A body is speeded up from 15 to 100 m.p.h. Reduce these speeds to feet per second. *Ans.* 22 ft./sec.; 146.7 ft./sec.

2. A man runs 100 yd. in 9.8 sec. Reduce this speed to feet per second and miles per hour. *Ans.* 30.61 ft./sec.; 20.88 m.p.h.

3. The equation of motion of a particle is $s = 16.1t^2$, s being in feet, and t in seconds. What is the velocity at the end of the sixth second? What is the average velocity during the sixth second? What is the average velocity during the last tenth of the sixth second?

Ans. 193.2 ft./sec.; 177.1 ft./sec.; 191.6 ft./sec.

4. The equation of motion of a particle is $s = -10 + 3t + 8t^2$, s being in feet, and t in seconds. Compute the initial velocity. Compute the velocity when the displacement is zero. *Ans.* 3 ft./sec.; 18.14 ft./sec.

97. Rectilinear Acceleration.—*Acceleration* is the rate of change of velocity. If the velocity is constant, the acceleration is of course zero. If the velocity is changed by equal amounts in equal time intervals, the acceleration is constant; if by unequal amounts in equal time intervals, it is variable. When the velocity increases, the acceleration is usually called positive; when the velocity decreases, the acceleration is usually called negative.

If the acceleration is constant, it is the total change in velocity during unit time, and its amount is obtained by dividing the total change in velocity by the time t in which the change was made. If a represents the acceleration, v_0 the initial velocity, and v the final velocity,

$$\mathbf{a} = \frac{\mathbf{v} - \mathbf{v}_0}{\mathbf{t}}$$

If $v_0 = 0$,

$$\mathbf{a} = \frac{\mathbf{v}}{\mathbf{t}}$$

If the acceleration is variable, its value at any instant is given by the ratio of the infinitesimal change in velocity dv to the corresponding time dt, or

$$\mathbf{a} = \frac{\mathbf{dv}}{\mathbf{dt}} = \frac{\mathbf{d^2s}}{\mathbf{dt^2}}$$

By eliminating dt between the equations $v = ds/dt$ and $a = dv/dt$, the important relation

$$\mathbf{v\,dv = a\,ds}$$

is obtained.

The units used are those of velocity and time. If the velocity of a body changes from 0 to 20 feet per second in 4 seconds, its

acceleration is a velocity change of 5 feet per second in a second, commonly written 5 ft./sec.^{2}. If the velocity of a body decreases from 40 to 20 miles per hour in 10 seconds, the acceleration is -2 miles per hour per second.

Acceleration, like velocity and displacement, is a vector quantity and is represented graphically by a vector.

Problems

1. An automobile is accelerated from a speed of 10 m.p.h. to one of 30 m.p.h. in 3 sec. Compute its average acceleration in feet per second per second. *Ans.* 9.78 ft./sec.^{2}.

2. The piston of a locomotive attains a speed of 20 ft./sec. with respect to the end of the cylinder in 0.038 sec. Compute the average acceleration. *Ans.* 526.32 ft./sec.^{2}.

3. The equation of motion of a particle is $s = 6t^2 + 4t^3$, s being in feet and t in seconds. Compute the initial acceleration. Compute the acceleration when the displacement is 100 ft. *Ans.* 12 ft./sec.^{2}; 72 ft./sec.^{2}.

98. Motion of Particle with Constant Acceleration.—The integration of the differential expression $a = dv/dt$ between the proper limits gives the velocity in terms of the time. A second integration gives the distance in terms of the time.

$$a = \frac{dv}{dt}, \text{ or } dv = a\,dt$$

Let v_0 be the initial velocity. Then, if a is constant,

$$\int_{v_0}^{v} dv = a\int_0^t dt$$

$$v - v_0 = at$$

or

$$\mathbf{v = v_0 + at} \tag{1}$$

Since

$$v = \frac{ds}{dt}, \quad ds = v_0\,dt + at\,dt$$

$$\int_0^s ds = v_0\int_0^t dt + a\int_0^t t\,dt$$

$$\mathbf{s = v_0t + \tfrac{1}{2}at^2} \tag{2}$$

The expression $v\,dv = a\,ds$ may be integrated in a similar manner.

$$\int_{v_0}^{v} v\,dv = a\int_0^s ds$$

$$\frac{v^2}{2} - \frac{v_0^2}{2} = as$$

or

$$v^2 = v_0^2 + 2as \tag{3}$$

If

$$v_0 = 0, \quad v^2 = 2as$$

Equations (1), (2), and (3) may be derived by simple algebra, as follows: If a particle gains a velocity of a during each unit of time, in t units of time it will have gained at units of velocity. Its final velocity, then, will be the sum of its initial velocity v_0 and the velocity it has gained, at.

$$v = v_0 + at \tag{1}$$

The average velocity during time t will be the mean of its initial velocity v_0 and its final velocity $v_0 + at$. The average velocity is $\dfrac{v + v_0}{2} = v_0 + \dfrac{1}{2}at$. The space passed over will be given by the product of the average velocity and the time, or

$$s = \frac{v + v_0}{2}t = v_0t + \frac{1}{2}at^2 \tag{2}$$

If t is eliminated from equations (1) and (2), the resulting equation is

$$v^2 = v_0^2 + 2as \tag{3}$$

Problems

1. A car traveling at a speed of 40 m.p.h. is brought to rest by means of the brakes in a distance of 100 ft. Compute the average acceleration.

Ans. −17.21 ft./sec.².

2. A particle has an initial speed of 8 ft./sec. and an acceleration of 5 ft./sec.². Compute its speed at the end of 3 sec. Compute its speed after it has moved 60 ft. Compute the distance the particle moves during the sixth second. *Ans.* 23 ft./sec.; 25.77 ft./sec.; 35.5 ft.

99. Falling Bodies, Air Resistance Neglected.—For comparatively short falls of bodies near the surface of the earth, the acceleration due to gravity may be considered constant if air resistance is neglected. The acceleration due to gravity is commonly denoted by g, and its value is approximately 32.2 feet per second per second. If no other value is given, this should be used in the solution of all problems involving falling bodies. (The accurate value is given by

$$g = 32.0894(1 + 0.0052375 \sin^2 l)(1 - 0.0000000957h)$$

in which l is the latitude in degrees, and h is the elevation above sea level in feet.)

The equations of motion derived in the preceding article become, for falling bodies,

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{gt} \tag{1}$$
$$\mathbf{h} = \mathbf{v}_0\mathbf{t} + \tfrac{1}{2}\mathbf{gt}^2 \tag{2}$$
$$\mathbf{v}^2 = \mathbf{v}_0^2 + \mathbf{2gh} \tag{3}$$

Space, velocity, and acceleration are all considered to be positive downward. If the body falls from rest, $v_0 = 0$; hence $v = \sqrt{2gh}$. If the body is projected vertically upward, v_0 is negative. The body rises $t = v_0/g$ seconds through a distance $h = -v_0^2/2g$ where it comes to rest. It then falls from rest and passes the initial point with a velocity of $+v_0$, continuing downward from that point exactly as though projected downward with the same velocity.

EXAMPLE

A ball is shot upward with a velocity of 30 ft./sec. One second later another is shot upward with a velocity of 100 ft./sec. Where and when do they pass?

Solution.—Let s_1 be the distance from the starting point to the point where they pass, t_1 the time elapsing after the first is discharged, and t_2 the time elapsing after the second is discharged. Then

$$t_2 = t_1 - 1$$

The initial velocity v_0 is negative in each case. After t_1 sec., the first ball will be a distance from the starting point $s = v_0t + \frac{1}{2}gt^2$, or

$$s_1 = -30t_1 + 16.1t_1^2$$

At the same instant, the second ball, which has been traveling only $t_1 - 1$ sec., will be a distance from the starting point $s = v_0t + \frac{1}{2}gt^2$, or

$$s_2 = -100(t_1 - 1) + 16.1(t_1 - 1)^2$$

When the two balls pass, $s_1 = s_2$, so

$$-30t_1 + 16.1t_1^2 = -100(t_1 - 1) + 16.1(t_1 - 1)^2$$
$$t_1 = 1.137 \text{ sec.}$$
$$s_1 = s_2 = -13.3 \text{ ft.}$$

It will be noticed that the first ball has reached the top point in its path in 30/32.2, or 0.932 sec., and is therefore moving downward when they pass.

Problems

1. Solve the example above if the velocity of the first ball is 60 ft./sec. upward. *Ans.* -55 ft.; 1.61 sec.

2. In the foregoing example how long after the first ball is shot upward must the second ball be discharged in order that they may pass at the upper point of the path of the first ball? *Ans.* 0.789 sec.

3. A ball drops freely from rest a distance of 2 ft. Compute the velocity with which it strikes and the time required. *Ans.* 11.35 ft./sec.; 0.352 sec.

4. Compute the height from which a ball must be dropped in order to strike with a velocity of 40 ft./sec. Compute the time required.
Ans. 24.84 ft.; 1.242 sec.

5. Compute the initial velocity required in order that a ball shall move downward a distance of 30 ft. in 0.5 sec. *Ans.* 51.95 ft./sec.

100. Composition and Resolution of Velocities and Accelerations.—Since velocity and acceleration are vector quantities, they may be combined into resultants or resolved into components the same as other vector quantities, such as forces and displacements. A velocity may be resolved into its x and y components; an acceleration, into its normal and tangential components.

Two component velocities of a particle may be combined vectorially to give the resultant velocity. Two or more component accelerations of a particle may be combined vectorially to give the resultant acceleration.

Problems

1. A car is moving down a 6 per cent grade with a velocity of 100 ft./sec. Get the horizontal and vertical components of the velocity.
Ans. 99.82 ft./sec.; 5.99 ft./sec.

2. A particle has an acceleration vertically downward of 32.2 ft./sec.^{2} and an acceleration horizontally of 44 ft./sec.^{2}. Compute the resultant acceleration. Compute the velocity of the particle after 0.2 sec. from rest.
Ans. 54.52 ft./sec.^{2}, 36°10′ with horizontal; 10.9 ft./sec., 36°10′ with horizontal.

101. Relative Motion.—By *velocity of a body* is usually meant the velocity of the body with respect to the point on the earth from which the motion is observed. Although any point on the earth has several motions in space, it is considered to be at rest, and the motion of any body relative to that point on the earth is called its *absolute velocity*. The velocity of one body with respect to another body is called its *relative velocity*.

Let A, Fig. 313, represent the top view of a flatcar, and B a body on the car. If the car A moves into the position A' in 1 second, BB_2 is the amount of its velocity and also its displacement. If the body B moves from the position B to B_1 relative

to the car while the car has moved from A to A', the vector BB', the resultant of BB_1 and BB_2, gives the absolute velocity and displacement of B.

Similarly, if vector BB_2 represents to some scale the absolute acceleration of the car A, and vector BB_1 represents the relative acceleration of B with respect to A, the vector BB' represents the absolute acceleration of B.

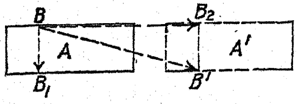


FIG. 313.

If either the absolute velocity or acceleration of A or the relative velocity or acceleration of B with respect to A is a variable, the absolute velocity or acceleration of B is given by the vector sum of the corresponding instantaneous values of the components.

This principle may be formulated as follows: The absolute displacement, velocity, or acceleration of any body plus (vectorially) the relative displacement, velocity, or acceleration of another body with respect to the first, equals the absolute displacement, velocity, or acceleration, respectively, of the second. Stated more briefly:

The absolute of A + the relative of B to A = the absolute of B.

If any two of these three quantities are known, the other may be found.

EXAMPLE 1

A man swims across a stream which flows at the rate of 2 ft./sec. If he can swim at the rate of 3 ft./sec., in what direction must he swim in order to land directly opposite? If the stream is 1000 ft. wide, find the time to cross.

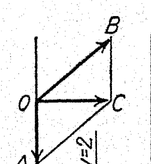


FIG. 314.

Solution.—If O, Fig. 314, is the point from which the swimmer starts, OC is the required direction of his absolute velocity. OA, two units to scale, represents the absolute velocity of the stream. Then OB, three units to scale, the vector representing the relative velocity of the swimmer with respect to the stream, must be at such an angle that their resultant lies along OC. The graphical construction gives angle $COB = 42°$, or, by trigonometry, sin COB must equal $\frac{2}{3}$, so $COB = 41°49'$.

Length OC is the absolute distance traveled by the swimmer in 1 sec. $OC = \sqrt{5} = 2.236$ ft. The time to cross is $t = 1000/2.236 = 447$ sec. $= 7$ min. 27 sec.

EXAMPLE 2

An ice boat runs due east with a velocity of 30 m.p.h. The wind blows from the northwest with a velocity of 20 m.p.h. How can the sail be set so that a forward pressure will be exerted?

Solution.—In this problem the two absolute velocities are given, to find the velocity of the wind relative to the boat. From O, Fig. 315, lay down the vectors representing the absolute velocities of the boat and the wind. Join the ends of the vectors with the line AB, and through O draw the vector OC, equal and parallel to AB. The vector OC represents to scale the velocity of the wind relative to the boat. Angle $COD = 41°40'$. If, now, the sail is set in some such position as MN, at an angle with the axis of the boat less than $41°40'$, the wind striking it in the direction OC will exert a small forward thrust. If this thrust is equal to the frictional resistance, the velocity will be maintained; whereas if it is greater, the velocity will be still further increased. It is thus seen that an ice boat may travel faster than the wind that drives it, a fact that has often been proved experimentally.

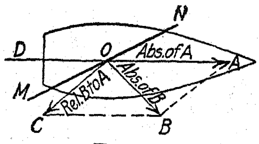


FIG. 315.

Problems

1. Water enters an inward-flow, radial-impulse turbine (Fig. 316) at an angle of 45° with the radius produced. If its velocity is 180 ft./sec., and the rim velocity of the wheel is 90 ft./sec., what should be the angle of the outer edge of the vane in order that the water may enter smoothly? What is the initial relative velocity? *Ans.* 16°20′ with the radius; 132.6 ft./sec.

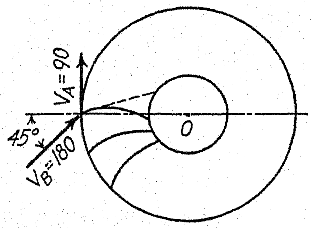


FIG. 316.

2. If the angle of the outer edge of the vane with the radius in Fig. 316 is 15°, and the velocity of the jet is 300 ft./sec. at an angle of 50° with the radius produced, what should be the speed of the rim of the wheel for smooth flow? What is the initial relative velocity? *Ans.* 178.13 ft./sec.; 199.3 ft./sec.

3. In Fig. 317, AB represents the connecting rod, and OB the crank of a reciprocating engine. Solve for the velocity of the crosshead A when the crankpin has a velocity of 20 ft./sec. and the crank is at an angle of 30° with the line AO. (The relative velocity of B with respect to A must necessarily be normal to AB, since AB is a rigid body.) *Ans.* 11.45 ft./sec.

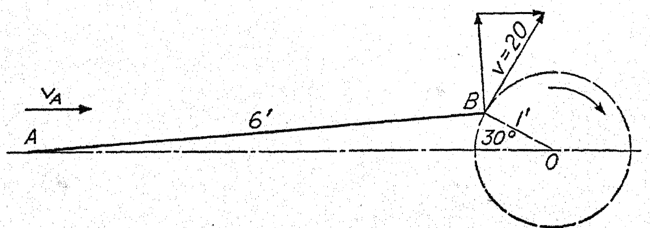


FIG. 317.

4. Solve for the velocity of the crosshead (Fig. 317) if angle AOB is 105°.
Ans. 18.47 ft./sec.

102. Displacement in Curvilinear Motion.—In this discussion, only *plane* curvilinear motion will be considered. Let curve $ABCD$, Fig. 318, represent the path of a particle that moves from point A through points B and C to point D. When the particle is at B, its displacement from its original position A is the vector AB. Likewise, when it is at C, its displacement is the vector AC; and when it is at D, its displacement is AD. Its displacement is independent of the path and depends only upon its original and final positions.

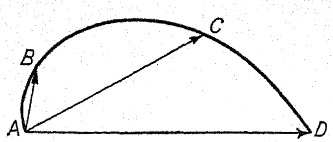


FIG. 318.

103. Velocity in Curvilinear Motion.—The *velocity* of a particle having curvilinear motion is the time rate of its displacement. If a particle moves along a curved path, as from A to B, Fig. 318, its displacement is the vector AB. If this displacement takes place in Δt time, the average velocity of the particle is given by the expression $\overline{AB}/\Delta t$. The direction of the vector AB is the average direction of the velocity, just as the quantity $\overline{AB}/\Delta t$ is the average amount of the velocity. As Δt is reduced in amount and becomes dt, approaching zero as a limit, point B will approach point A as a limit, so the limiting direction of the vector AB is the tangent at A. The direction of the instantaneous velocity at A is therefore the tangent to the path at A. As point B approaches point A, the vector AB and the arc AB approach equality, the arc AB becomes ds, and the magnitude of the instantaneous velocity at A is given by the expression $v = ds/dt$.

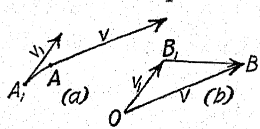


FIG. 319.

104. Acceleration in Curvilinear Motion.—Acceleration is the time rate of change of velocity. In curvilinear motion the velocity necessarily changes continually in direction and may also change in amount. Let the particle move along the curved path A_1A, Fig. 319(a), in time Δt, and let v_1 and v be the instantaneous velocities at A_1 and A, respectively. If from any point O, Fig. 319(b), vector OB_1 is laid off equal and parallel to v_1, and vector OB is laid off equal and parallel to v, vector B_1B represents the total change in velocity, and $B_1B/\Delta t$ gives the average *rate of change of velocity*, or *acceleration*, between A_1 and A. The limiting value of the average acceleration as Δt becomes dt is the *instantaneous acceleration* at A.

105. Tangential Acceleration and Normal Acceleration.—It is seen from Art. 104 that, in general, the acceleration is not in the direction of the velocity of the particle. Since the direction of the velocity is usually the direction of reference, the resultant acceleration a is usually determined by means of its two components a_t and a_n. The component a_t is the *tangential* component of the acceleration and is parallel to the direction of the velocity. The component a_n is the *normal* component of the acceleration and is perpendicular to the direction of the velocity.

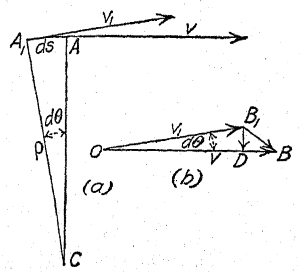


FIG. 320.

If A_1 and A, Fig. 320(a), are consecutive points in the path of the particle, the distance A_1A being the distance ds that is traversed in time dt, then B_1B, Fig. 320(b), is the change in velocity in time dt. The average acceleration is B_1B/dt and is the instantaneous acceleration at A, since dt approaches zero as a limit.

The acceleration B_1B/dt is resolved into its tangential and normal components; $DB/dt = a_t$, parallel to the direction of the velocity at A; and $B_1D/dt = a_n$, normal to the direction of the velocity at A. Since A_1 and A are consecutive points in the path, $DB = v - v_1 = dv$, so

$$\mathbf{a}_t = \frac{\mathbf{dv}}{\mathbf{dt}}$$

It will be seen that this is the rate of change of speed at point A. In the limit,

$$B_1D = v_1\,d\theta = v\,d\theta$$

Also,

$$d\theta = \frac{ds}{\rho} \text{ and } \frac{ds}{dt} = v$$

Then

$$a_n = \frac{B_1D}{dt} = \frac{v\,d\theta}{dt} = \frac{v\,ds}{\rho\,dt} = \frac{v^2}{\rho}$$

$$\mathbf{a}_n = \frac{\mathbf{v}^2}{\rho}$$

The expressions $a_t = dv/dt$ and $a_n = v^2/\rho$ are important ones and should be kept carefully in mind.

GENERAL PROBLEMS ON KINEMATICS OF A PARTICLE

1. The equation of motion of a particle is $s = -4 - 3t^2 + 2t^3$, s being in feet, and t in seconds. Compute the displacement and the acceleration when the velocity is zero. Compute the displacement and the velocity when the acceleration is zero. Compute the velocity and the acceleration when the displacement is zero.

Ans. −5 ft.; 6 ft./sec.^{2}; −4.5 ft.; −1.5 ft./sec.; 12 ft./sec.; 18 ft./sec.^{2}.

2. An elevator starts from rest and attains a velocity of 20 ft./sec. upward in a distance of 30 ft. with uniform acceleration, then moves uniformly upward at 20 ft./sec. a distance of 600 ft., then comes to rest with a uniform negative acceleration in a distance of 16 ft. Compute the total time required. *Ans.* 34.6 sec.

3. In an elevator shaft 300 ft. high, an elevator is moving upward with a constant velocity of 14 ft./sec. At the instant it is 20 ft. from the bottom, a ball is dropped from the top of the shaft. Compute the time until the ball and elevator meet, the distance from the bottom of the shaft, and the relative velocity. *Ans.* 3.76 sec.; 72.7 ft.; 135 ft./sec.

4. From the top of a tower 400 ft. high, a ball is projected downward with an initial velocity of 20 ft./sec. at the same instant that another is projected upward from the bottom with a velocity of 140 ft./sec. Compute the distance from the bottom of the tower at which they pass. Prove that their relative velocity is always 160 ft./sec. *Ans.* 249.4 ft.

5. A ball is projected upward with an initial velocity of 120 ft./sec. One second later, another is projected upward with the same initial velocity. Compute the time after the first ball is discharged until the two balls pass, and the distance above the point of discharge. Compute their relative velocity, and prove that it is constant.

Ans. 4.227 sec.; 219.64 ft.; 32.2 ft./sec.

6. From what height above the top of a pile must the hammer of a pile driver fall in order to strike the pile with a velocity of 10 ft./sec.?

Ans. 1.56 ft.

7. An automobile accelerates from a speed of 15 to one of 45 m.p.h. in a distance of 500 ft. Compute the average acceleration and the time required.

Ans. 3.872 ft./sec.^{2}; 11.38 sec.

8. A train is brought to rest from a speed of 60 m.p.h. in a distance of 4000 ft. Compute the average acceleration and the time required.

Ans. −0.968 ft./sec.^{2}; 91 sec.

9. An airplane is headed straight north and is moving through the air at a speed of 120 m.p.h. The wind is coming from a point 30° north of west with a velocity of 30 m.p.h. with respect to the earth. Compute the direction and velocity of the airplane with respect to the earth.

Ans. N 13°55′ E; 108.2 m.p.h.

10. With the same wind referred to in Prob. 9, an airplane is headed toward a fixed point 80 miles away in a direction N 60° E at the same speed (120 m.p.h.) with respect to the air. How long will it require for the air-

plane to reach the fixed point? From what direction does the wind appear to come? *Ans.* 36.4 min.; N 47°30′ E.

11. An automobile is accelerated from a speed of 15 m.p.h. to one of 50 m.p.h. in 13 sec. Compute the average acceleration and the distance that it travels. *Ans.* 3.95 ft./sec.^{2}; 619.7 ft.

12. A racing car is brought to rest from a speed of 300 m.p.h. in a distance of 5 miles. Compute the average acceleration and the time required *Ans.* -3.67 ft./sec.^{2}; 2 min.

CHAPTER XII

KINETICS OF RIGID BODIES IN RECTILINEAR TRANSLATION

106. Definitions and General Principles.—A *rigid body* is a system of particles that always have a fixed relation to each other.

The motion of a rigid body is said to be *rectilinear translation* if the motion of each particle of the body is along a straight line and parallel to the line of motion of each of the other particles of the body. The motion of any particle of the body may therefore be taken as representing the motion of the entire rigid body. The entire rigid body may be treated as though it were a particle having the same position and motion as the center of gravity of the body.

The motion of the body of a railway car along a straight track, the motion of the piston of a stationary reciprocating engine in its cylinder, and the motion of the hammer of a pile driver in its guides are all examples of rectilinear translation.

107. Newton's Three Laws of Motion.—Sir Isaac Newton formulated the following *laws of motion* for a particle, generalized from observation:

1. A particle remains at rest or continues in uniform motion in a straight line unless it is acted upon by a resultant force.
2. A particle acted upon by a resultant force receives an acceleration in the direction of the force that is proportional to the force and inversely proportional to the mass of the particle.
3. For every action upon a particle, it exerts an equal, opposite, and collinear reaction.

108. Relation between Force, Mass, and Acceleration.—Newton's second law of motion (Art. 107) states that the accelerations of bodies are directly proportional to the resultant forces acting and inversely proportional to the masses acted upon. Let F be the resultant force that acts upon mass M to produce acceleration a. Then a varies as F/M, or F varies as Ma.

$$F = KMa$$

K being a constant, the value of which depends upon the units used. In American engineering practice the unit of force used is the pound, and the unit of acceleration is the foot per second per second. In order to make the constant $K = 1$ and thus simplify the expression, the unit of mass used is that amount in which unit force produces unit acceleration. If a resultant force of 1 pound acts upon a quantity of matter weighing 1 pound, the acceleration produced is 32.2 feet per second per second, as in the case of a falling body. If the quantity of matter is increased, and the force remains constant, the acceleration will decrease proportionately, so that, if the quantity of matter weighs 32.2 pounds, the force of 1 pound will produce an acceleration of 1 foot per second per second. It is seen, then, that 32.2 pounds of matter is the unit of mass in which unit force produces unit acceleration; and if this unit is used, $K = 1$. In order to obtain the number of mass units in a given quantity of matter, its weight in pounds must be divided by 32.2. That is, $M = W/g$. Then

$$\mathbf{F} = \mathbf{Ma} = \frac{\mathbf{W}}{\mathbf{g}}\mathbf{a}$$

In the equation $F = Ma$, M is a scalar quantity, and F and a are vector quantities. If F_1 and F_2 are any two components into which the force F may be resolved, and a_1 and a_2 are the corresponding components of the acceleration, parallel, respectively, to F_1 and F_2, it follows that

$$F_1 = Ma_1 \text{ and } F_2 = Ma_2$$

If the components are the rectangular components F_x and F_y,

$$F_x = Ma_x \quad \text{and} \quad F_y = Ma_y$$

EXAMPLE 1

A horizontal force of 10 lb. is exerted upon a body whose weight is 100 lb. and which is resting upon a smooth horizontal surface. What is the velocity of the body at the end of 5 sec., and what is the distance passed over?

Solution.

$$F = Ma = \frac{W}{g}a$$

$$10 = \frac{100}{32.2}a$$

$$a = 3.22 \text{ ft./sec.}^2$$

The force is constant, so the body has uniformly accelerated motion. Since the body starts from rest, the equations of motion are

$$v = at, \text{ and } s = \tfrac{1}{2}at^2$$
$$v = 3.22 \times 5 = 16.1 \text{ ft./sec.}$$
$$s = \tfrac{1}{2} \times 3.22 \times 25 = 40.25 \text{ ft.}$$

EXAMPLE 2

A horizontal force of 80 lb. is applied to a body weighing 150 lb. to push it up a 15° plane, as shown in Fig. 321(*a*). If $f = 0.1$, what will be the acceleration of the body?

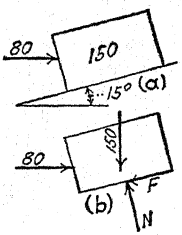


FIG. 321.

Solution.—In Fig. 321(*b*) is shown the free-body diagram. The forces acting upon the body are four in number: its own weight, 150 lb.; the horizontal force of 80 lb.; the normal reaction N of the plane; and the frictional resistance F of the plane.

Since the body is in equilibrium normal to the plane the normal reaction N is given by the equation

$$N = 150 \cos 15° + 80 \sin 15°$$
$$N = 165.6 \text{ lb.}$$
$$F = fN = 16.56 \text{ lb.}$$

The summation of forces parallel to the plane gives

$$\Sigma F = 80 \cos 15° - 150 \sin 15° - 16.56$$
$$\Sigma F = 21.84 \text{ lb.}$$
$$\Sigma F = Ma = \frac{W}{g}a$$
$$21.84 = \frac{150}{32.2}a$$
$$a = 4.69 \text{ ft./sec.}^2$$

Problems

1. A resultant force of 5 lb. acts for 20 sec. upon a 200-lb. body. Compute the final velocity and the distance moved if the body starts from rest.
Ans. 16.1 ft./sec.; 161 ft.

2. An elevator weighing 2000 lb. is given a velocity of 12 ft./sec. upward in 3 sec. with uniform acceleration. Compute the distance moved and the tension in the supporting cables. *Ans.* 18 ft.; 2248 lb.

3. Solve for the tension in the cables of the elevator described in Prob. 2 if the tension is reduced so that the elevator comes to rest from its velocity of 12 ft./sec. in a distance of 12 ft. Compute also the time required.
Ans. 1627 lb.; 2 sec.

4. A block is projected up a 30° plane for which $f = 0.3$ with an initial velocity of 30 ft./sec. Compute its position and its velocity after 1 sec.; after 3 sec.

Ans. 17.77 ft., 5.53 ft./sec.; 6.33 ft., −13.7 ft./sec.

109. Effective Forces on a Rigid Body: D'Alembert's Principle. In general, any particle of a body considered free has a system of

forces acting upon it, some of which may be external to the body as a whole, and some of which are internal. The resultant of all these forces for the particle is called the *effective force* for the particle and is equal to $dM \cdot a$, dM being the mass of the particle, and a its acceleration. If the particles of the body were all made free of each other, and each had its effective force acting, the motion of the system of particles would be the same as the actual motion of the body. The resultant of all these effective forces for all the particles of the body is called the *resultant effective force* for the body.

Since the internal forces between the particles of a rigid body are always mutual, that is, equal and opposite, their total resultant for the whole body is zero. It follows, then, that the *resultant effective force* for all the particles of a rigid body must be equivalent to the resultant of the *external* forces. If F is the resultant of the external forces,

$$F = \int dMa$$

If the motion is translation, a is the same in amount and direction for all the particles, so, for translation, $F = a\Sigma\, dM = Ma$.

Since each particle has a force equivalent to $dM \cdot a$ acting upon it, and since each force is proportional to the mass of the particle, the point of application of the resultant is necessarily the same as that of a system of particles acted upon by their own weights. As shown in Art. 71, this is the mass center of the body.

Since the system of effective forces upon the particles of a rigid body could replace the actual force system acting upon the body, it follows that if this system of effective forces were reversed and added to the actual force system, the result would be equilibrium of the body without changing any of the actual external forces.

This is called D'Alembert's principle and is applicable to both rigid and nonrigid bodies, but only in the case of rigid bodies is it sufficient to determine the motion.

By this method a problem in kinetics is reduced to a simpler one in statics, for then all the equations of static equilibrium will apply: $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$. The student should keep in mind that this is only an *imaginary* force system, added to the actual system for the purpose of solution.

This method of procedure does not conflict with the method of Art. 108, as will now be shown. In Fig. 322(a), let F be the

resultant force acting upon mass M. From Art. 108, the force F produces an acceleration a in the mass M of such an amount that

$$F = Ma = \frac{W}{g}a$$

In Fig. 322(b), F is the resultant force acting upon mass M. The resultant *effective force* $\frac{W}{g}a$ is reversed and added to the system

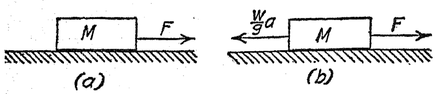


FIG. 322.

to produce a condition of equilibrium. Since the system of forces acting upon the mass M is now in equilibrium, $\Sigma F_x = 0$, so

$$F - \frac{W}{g}a = 0, \text{ or } F = \frac{W}{g}a$$

as before.

110. Reactions on Accelerated Body.—If an unbalanced system of forces is acting upon a rigid body to produce rectilinear acceleration, and if the resultant effective force Ma is added to the actual force system in the direction opposite to the direction of the acceleration, and acting through the center of gravity of the body, the combined systems are in equilibrium, as explained in Art. 109. By this means, the moment equation $\Sigma M = 0$ may be written with respect to any convenient point in order to determine any unknown reactions upon an accelerated rigid body. (It should be noted that, if the reversed effective force is not used, and if the X axis is taken in the direction of the acceleration, $\Sigma F_x = Ma$, and $\Sigma F_y = 0$. The equation $\Sigma M = 0$ can be written only for some point on a line through the center of gravity in the X direction.)

EXAMPLE

A safe with weight and dimensions as shown in Fig. 323(a) is pulled along a horizontal track by a force of 100 lb. A force of 60 lb. is sufficient to move it uniformly. Determine the normal components of the reactions at A and B.

Solution.—Figure 323(b) shows the free-body diagram, with all the external forces acting and in addition the *reversed effective force* $\frac{W}{g}a$ acting

through the center of gravity. The free body now has a balanced system of forces acting, and the equations of equilibrium are true.

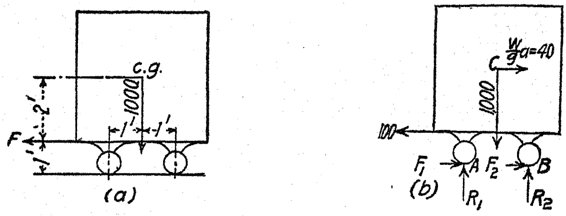


FIG. 323.

Since 60 lb. will move the body uniformly, $F_1 + F_2 = 60$.

$$\Sigma F_x = \frac{W}{g}a$$

$$100 - 60 = \frac{W}{g}a = 40$$

The equation $\Sigma M_A = 0$ gives

$$(100 \times 1) - (40 \times 3) - (1000 \times 1) + (R_2 \times 2) = 0$$
$$R_2 = 510 \text{ lb.}$$

From the equation $\Sigma F_y = 0$,

$$R_1 = 490 \text{ lb.}$$

Problems

1. Solve the foregoing example if the 100-lb. force is applied at the upper corner of the safe 2 ft. above the center of gravity.
Ans. $R_1 = 690$ lb.; $R_2 = 310$ lb.

2. A car and its load weigh 4000 lb., and the center of gravity is 26 in. from the ground and midway between the front and rear wheels which are 122 in. apart. The car is brought to rest from a speed of 40 m.p.h. in 6 sec. by means of the brakes. Compute the normal pressures on the front wheels and on the rear wheels. *Ans.* 2260 lb.; 1740 lb.

3. If the car described in Prob. 2 is accelerated from a speed of 30 m.p.h. to one of 80 m.p.h. in a distance of 1800 ft., what are the normal pressures on the front and rear wheels? *Ans.* 1910 lb.; 2090 lb.

4. A part of a stone column is 2 ft. in diameter and 10 ft. high. If hauled on a truck standing in a vertical position, what is the limiting acceleration that may be given to the truck before tipping impends if the truck is on a level roadway? What is the limiting negative acceleration if the truck is going down a 5 per cent grade? *Ans.* 6.44 ft./sec.^{2}; 4.824 ft./sec.^{2}.

GENERAL PROBLEMS ON KINETICS OF RIGID BODIES IN RECTILINEAR TRANSLATION

1. A pile-driver hammer weighing 1200 lb. drops 10 ft. upon the head of a pile. If its actual velocity of striking is 24 ft./sec., what is the total air and guide resistance, assuming it to be constant? *Ans.* 127 lb.

2. The hammer of the pile driver described in Prob. 1 is drawn back up at a constant speed of 4 ft./sec. If this speed is gained in 0.4 sec. by means of the clutch, and air and guide resistance is assumed to be 100 lb., what is the tension in the supporting cable and the distance the hammer moves till it gains full speed? *Ans.* 1673 lb.; 0.8 ft.

3. Two blocks are connected by a cord over a pulley as shown in Fig. 324. Neglecting the mass of the pulley and cord and the friction at the pulley, compute the tension in the cord and the time for the blocks to move 10 ft. from rest if the coefficient of friction $f = 0.4$.

Ans. 76.4 lb.; 1.31 sec.

4. Solve Prob. 3 if the 100-lb. block is being pulled up a 60° plane by the 120-lb. block. *Ans.* 112.7 lb.; 3.2 sec.

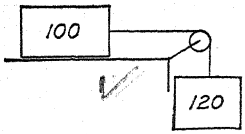


FIG. 324.

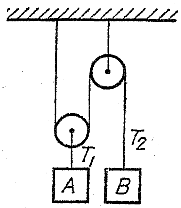


FIG. 325.

5. An automobile is advertised as being able to accelerate from a speed of 10 m.p.h. to one of 50 m.p.h. in 17.7 sec. Compute its acceleration and the distance required. *Ans.* 3.325 ft./sec.²; 779 ft.

6. In Fig. 325, let body A weigh 100 lb. and body B weigh 60 lb. Neglecting the mass of the pulleys and cords and the friction at the pulleys, compute tensions T_1 and T_2 and the acceleration of each body.

Ans. 105.88 lb.; 52.94 lb.; 1.89 ft./sec.²; 3.78 ft./sec.².

7. Solve Prob. 6 if weight A is 150 lb. and all other data remain the same.

Ans. 138.46 lb.; 69.23 lb.; 2.48 ft./sec.²; 4.96 ft./sec.².

8. The three bodies A, B, and C, Fig. 326, weigh 1, 2, and 3 lb., respectively. If they are released from rest in the position shown, what will be the tension in each cord and the velocity of each after 1 sec.? Neglect the friction and the mass of the cords and pulleys.

Ans. 1.41 lb.; 2.82 lb.; 13.26 ft./sec.; 9.47 ft./sec.; 1.895 ft./sec.

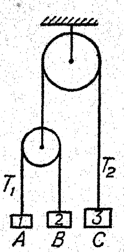


FIG. 326.

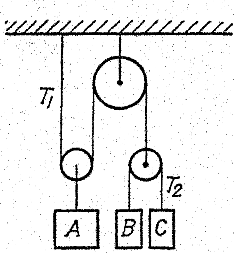


FIG. 327.

9. In Fig. 327, body A weighs 200 lb., body B weighs 60 lb., and body C weighs 40 lb. If they are released from rest in the position shown, what

will be the tension in each cord and the distance that each body moves in 1 sec.? Neglect the mass of the cords and pulleys and the friction at the pulleys. *Ans.* 98.6 lb.; 49.3 lb.; 0.22 ft.; 2.865 ft.; 3.75 ft.

10. The two blocks shown in Fig. 328 are free to slide down the 30° plane. They start from rest at M and reach N, 40 ft. from M, with a velocity of 25 ft./sec. The coefficient of friction under B is 0.3. Compute the coefficient of friction under A, the pressure between the blocks, and the time for the blocks to move from M to N. *Ans.* 0.294; 0.029 lb.; 3.2 sec.

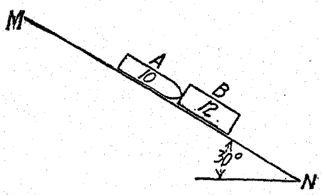


FIG. 328.

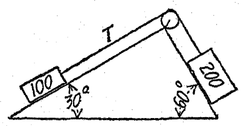


FIG. 329.

11. If the coefficient of friction under the blocks shown in Fig. 329 is $\frac{1}{3}$, compute the acceleration of the blocks, the tension in the cord, and the time for the blocks to move 8 ft. from rest.

Ans. 6.55 ft./sec.²; 99.2 lb.; 1.56 sec.

12. In Fig. 330, the coefficient of friction under the 80-lb. block is 0.4. Solve for the weight W necessary to give the 80-lb. block an acceleration of 6 ft./sec.² up the plane. Solve also for the tension in the cord and for the velocity of the 80-lb. block after it moves 10 ft. from rest.

Ans. 207.5 lb.; 94.09 lb.; 10.95 ft./sec.

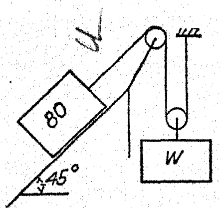


FIG. 330.

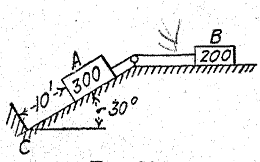


FIG. 331.

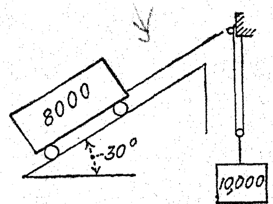


FIG. 332.

13. Solve Prob. 12 if the acceleration of the 80-lb. block is 6 ft./sec.² *down* the plane. *Ans.* 34.8 lb.; 19.03 lb.; 10.95 ft./sec.

14. In Fig. 331, $f = 0.2$ under both blocks. If the blocks are released from rest in the position shown, get the tension in the cord and the velocity of the 300-lb. block as it strikes the stop at C. Get also the distance that block B moves after block A comes to rest.

Ans. 63.2 lb.; 8.65 ft./sec.; 5.81 ft.

15. The empty ore car shown in Fig. 332 is being pulled up the incline by means of the counterweight of 10,000 lb. If car resistance is 40 lb. per ton, compute the acceleration of the car and the tension in the cable. What is the time required for the car to move 120 ft. from rest?

Ans. 2.576 ft./sec.²; 4800 lb.; 9.65 sec.

16. How much ore may be loaded into the car described in Prob. 15 if the velocity at the foot of the incline is not to exceed 10 ft./sec.?

Ans. 2775 lb.

17. A railway car starts from rest on a 0.75 per cent grade and runs down under the influence of gravity a distance of 3000 ft. From there the track is level. Train resistance is assumed constant at 8 lb. per ton. Solve for the velocity at the foot of the grade, the time till the car comes to rest, and the total distance that it travels.

Ans. 26 ft./sec.; $t_1 = 231$ sec.; $t_2 = 202$ sec.; 5625 ft.

18. A railway car is released from rest at the top of a 2 per cent grade 2000 ft. long. At the foot of the grade is a level stretch of track 1600 ft. long, then an upgrade of 1 per cent. Car resistance is considered constant at 10 lb. per ton. Where will the car come to rest the first time? Where will it finally come to rest? *Ans.* 5067 ft.; 2133 ft.

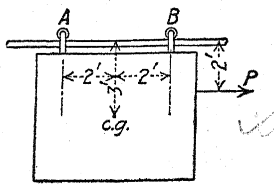


FIG. 333.

19. A railway car is started up a 1 per cent grade with a velocity of 15 m.p.h. If car resistance is 8 lb. per ton, how far up the grade will it go? If the car is then allowed to run back, what will be its velocity at the foot of the grade? If the track is then level, how far from the foot of the grade will it run? *Ans.* 535 ft.; 9.8 m.p.h.; 805 ft.

20. The door shown in Fig. 333 weighs 300 lb. and is hung from a track by means of wheels at A and B. Wheel B rolls, but wheel A slides on the track, f being 0.25. Solve for the amount of force P applied as shown to give the door an acceleration of 5 ft./sec.². Solve also for the reactions at A and B. *Ans.* $P = 86.1$ lb.; $A_Y = 158$ lb.; $A_X = 39.5$ lb.; $B = 142$ lb.

21. Solve Prob. 20 if force P is acting 3 ft. below the center of gravity of the door, both wheels are sliding, and the acceleration is 8 ft./sec.². The track can hold either up or down on the wheels.

Ans. $P = 186.25$ lb.; $A_Y = 373.5$ lb.; $A_X = 93.4$ lb.; $B_Y = -73.5$ lb.; $B_X = 18.4$ lb.

22. If in Fig. 333 force P is applied on a line 2 ft. below the center of gravity of the door, what is the maximum acceleration that may be given the door without reversing the direction of the reaction at B?

Ans. 12.075 ft./sec.².

23. A block 2 ft. square and 8 ft. long stands on end on a truck with its sides parallel to the direction of motion of the truck. The coefficient of friction $f = 0.275$. As the acceleration of the truck is increased, will the block slide or tip first, and for what value of the acceleration?

Ans. Tip; 8.05 ft./sec.².

24. Solve Prob. 23 if the sides of the block make an angle of 30° with the direction of motion. *Ans.* Slide; 8.85 ft./sec.².

CHAPTER XIII

MOMENT OF INERTIA OF MASSES

111. Definitions and Units.—The moment of inertia of the mass of a body with respect to any axis is the sum of the products of each elementary mass and the square of its distance from the axis. It is represented by I.

If M is the mass of a body, V its volume, γ the mass per unit volume, and ρ the distance of the elementary mass from the axis,

$$M = \gamma V \text{ and } dM = \gamma\, dV.$$

$$I = \int \rho^2\, dM = \gamma \int \rho^2\, dV$$

The radius of gyration of a body with respect to any axis is the distance from the axis at which all the mass of the body could be concentrated and have the same moment of inertia. It is represented by k.

$$I = Mk^2$$

From this equation it is seen that k^2 is the mean or average value of the variable quantity ρ^2 for equal values of dM.

The moment of inertia of the mass of a body is the product of a mass and a length squared, Mk^2 in its simplest form. Since the unit of mass commonly used in engineering is one containing g units of weight, and g is used in units of feet per second per second, all dimensions should be in feet. If the units of the quantities entering into I are used, it becomes

$$\text{Pounds} \times \text{seconds}^2 \times \text{feet}$$

In testing equations in which I enters in order to determine if they are homogeneous or not, the unit of I must be used as above. No name has been given to the unit of moment of inertia of mass, and commonly no unit is specified.

Problem

Compute γ for wood, 40 lb./cu. ft.; aluminum, 165 lb./cu. ft.; cast iron, 450 lb./cu. ft.; steel, 490 lb./cu. ft.; copper, 556 lb./cu. ft.

Ans. 1.24; 5.13; 14.0; 15.2; 17.3.

112. Moment of Inertia of Thin Plates.—Let Fig. 334 represent any thin plate, and OX and OY coordinate axes in the central plane of the plate, the Z axis being normal to the plate at O. Let dM be a differential mass of the plate, at distance x from the Y axis, y from the X axis, and ρ from O. Then, by definition, $I_X = \int y^2\,dM$, $I_Y = \int x^2\,dM$, and $I_Z = \int \rho^2\,dM$. Since $\rho^2 = x^2 + y^2$,

$$I_Z = I_X + I_Y$$

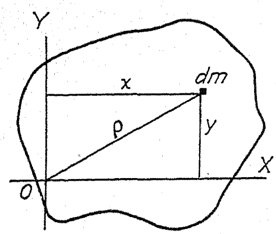


FIG. 334.

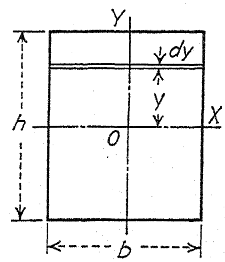


FIG. 335.

EXAMPLE 1

Derive the expressions for I_X, I_Y, and I_Z of the thin rectangular plate shown in Fig. 335.

Solution.—Let t be the thickness of the thin plate. The differential mass

$$dM = \gamma tb\,dy.$$

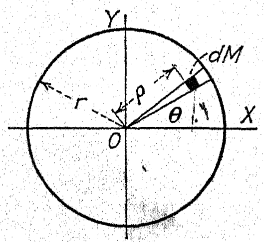


FIG. 336.

$$I_X = \int y^2\,dM$$

$$= \gamma tb \int_{-\frac{h}{2}}^{+\frac{h}{2}} y^2\,dy$$

$$= \tfrac{1}{3}\gamma tby^3 \Big]_{-\frac{h}{2}}^{+\frac{h}{2}}$$

$$= \tfrac{1}{12}\gamma tbh^3$$

Since $M = \gamma tbh$,

$$I_X = \tfrac{1}{12}Mh^2$$

Similarly,

$$I_Y = \tfrac{1}{12}Mb^2$$

Since $I_Z = I_X + I_Y$,

$$I_Z = \tfrac{1}{12}M(b^2 + h^2)$$

EXAMPLE 2

Derive the expressions for I_X, I_Y, and I_Z of the thin circular plate shown in Fig. 336.

Solution.—In this case, I_Z will be obtained first. The differential mass $dM = \gamma t\, dA$.

$$\begin{aligned} I_Z &= \int \rho^2\, dM \\ &= \gamma t \int_0^r \rho^2\, dA\, . \\ &= \tfrac{1}{2}\gamma t \pi r^4 \end{aligned}$$

$= \gamma t \int_0^r \int_0^{2\pi} \rho^2 \, d\rho \, \rho \, d\theta \, .$

$= 2\pi \gamma t \int_0^r \rho^3 \, d\rho .$

Since $M = \gamma t \pi r^2$,

$$I_Z = \tfrac{1}{2}Mr^2$$

Since, by symmetry, $I_X = I_Y$, and since $I_Z = I_X + I_Y$,

$$I_X = I_Y = \tfrac{1}{4}Mr^2$$

EXAMPLE 3

Derive the expressions for I_X, I_Y, and I_Z of the thin elliptic plate shown in Fig. 337.

Solution.—The equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, from which $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$. The equation of the circumscribed circle is $x^2 + y^2 = a^2$, from which $y = \sqrt{a^2 - x^2}$. It will be seen, then, that the differential area MN of the ellipse is b/a times the differential area $M'N'$ of the circle.

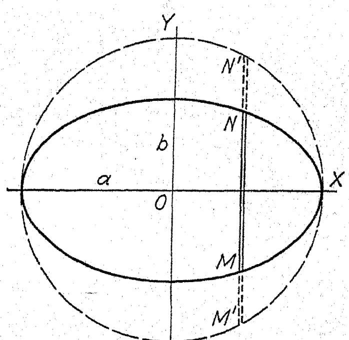


Fig. 337.

In the derivation of Example 1, it is seen that the moment of inertia of a rectangular thin plate is proportional to the cube of the height. It is evident, then, that the moment of inertia of the differential area MN will be b^3/a^3 times the moment of inertia of the differential area $M'N'$. Therefore, in the summation, the moment of inertia of the elliptic plate will be b^3/a^3 times the moment of inertia of the circular plate. For the circular plate,

$$I_X = \tfrac{1}{4}Ma^2 = \tfrac{1}{4}\gamma t \pi a^4$$

For the elliptic plate,

$$\begin{aligned} I_X &= \frac{1}{4}\gamma t \pi a^4 \times \frac{b^3}{a^3} \\ &= \gamma t \pi ab \times \frac{b^2}{4} \\ &= \tfrac{1}{4}Mb^2 \end{aligned}$$

Similarly,

$$\begin{aligned} I_Y &= \tfrac{1}{4}Ma^2 \\ I_Z &= I_Y + I_X = \tfrac{1}{4}M(a^2 + b^2) \end{aligned}$$

Problems

1. Derive the expression for I_X of a thin rectangular plate with base b and height h with respect to its base. *Ans.* $I_X = \frac{1}{3}Mh^2$.

2. Derive the expression for I_X of a thin triangular plate with base b and height h with respect (1) to its base; (2) to a centroidal axis parallel to the base; and (3) to an axis through the vertex parallel to the base.

Ans. $\frac{1}{6}Mh^2$; $\frac{1}{18}Mh^2$; $\frac{1}{2}Mh^2$.

3. Prove that the moment of inertia of a thin square plate with respect to any centroidal axis in its central plane is a constant.

4. By using Fig. 336, derive the expression for I_X directly by integrating $y^2\,dM$.

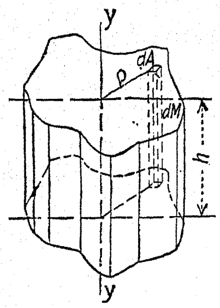


FIG. 338.

113. Moment of Inertia of Some Geometric Solids.—The moment of inertia of mass is used in problems in rotation and must be computed with respect to the axis of rotation. The axis of rotation of a body is usually a geometric axis or one parallel to it. As examples, the expressions for the moments of inertia and the radii of gyration of a number of regular-shaped bodies will now be obtained.

EXAMPLE 1

Show that for a right prism of altitude h, with respect to any axis perpendicular to the base,

$$I = \gamma h \times \text{polar } I \text{ of base}$$

with respect to the same axis.

Solution.—In Fig. 338 let the Y axis be any axis perpendicular to the base of the right prism shown. The elementary prism has an altitude h, a base dA, and a mass $dM = \gamma h\,dA$.

$$I_Y = \int \rho^2\,dM = \gamma h \int \rho^2\,dA$$

The quantity $\int \rho^2\,dA$ is the polar moment of inertia of the base with respect to the Y axis. Therefore,

$$I_Y = \gamma h \times \text{polar } I \text{ of base}$$

EXAMPLE 2

Derive the expressions for the moment of inertia and radius of gyration of a right circular cylinder with respect to its geometric axis.

Solution.—The base of the cylinder is a circle, and the geometric axis of the cylinder passes through the center of the circle. From Example 1,

$$I_Y = \gamma h \times \text{polar } I_Y \text{ of base}$$
$$I_Y = \gamma h \times \tfrac{1}{2}\pi r^4$$
$$I_Y = (\gamma \pi r^2 h) \times \tfrac{1}{2} r^2$$
$$I_Y = \tfrac{1}{2} M r^2$$
$$k = \sqrt{\frac{I}{M}} = \frac{r}{\sqrt{2}}$$

It should be noted that the expression for the moment of inertia of a cylinder with respect to its geometric axis may be obtained also from the result of Example 2, Art. 112.

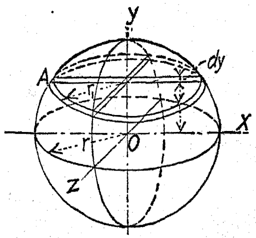


FIG. 339.

EXAMPLE 3

Derive the expressions for the moment of inertia and radius of gyration of a homogeneous sphere of radius r with respect to a diameter.

Solution.—In Fig. 339 let the Y axis be the inertia axis. Let the sphere be divided into thin plates by planes perpendicular to the Y axis, each of thickness dy and of radius r_1. One plate is shown at A.

$$r_1^2 = r^2 - y^2$$

and

$$dM = \gamma \pi r_1^2 \, dy = \gamma \pi \, (r^2 - y^2) dy$$

By Example 2, the moment of inertia of this thin plate with respect to the Y axis is

$$I_Y = \tfrac{1}{2} dM r_1^2 = \tfrac{1}{2} \gamma \pi (r^2 - y^2)^2 \, dy$$

If these differential moments of inertia are summed between the limits $-r$ and $+r$, the entire moment of inertia of the sphere is obtained.

$$I = \tfrac{1}{2}\gamma\pi \int_{-r}^{+r} (r^2 - y^2)^2 \, dy$$
$$I = \tfrac{2}{5} M r^2, \text{ since } M = \gamma V = \tfrac{4}{3}\gamma \pi r^3$$
$$k = \sqrt{\frac{I}{M}} = r\sqrt{\frac{2}{5}}$$

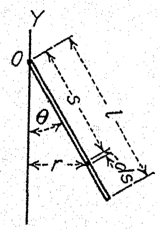


FIG. 340.

EXAMPLE 4

Derive the expression for the moment of inertia of a slender rod with respect to an axis through one end.

Solution.—Let l be the length of the rod (Fig. 340), W its weight, M its mass, A its cross-sectional area, and θ its angle with the Y axis.

$$dM = \gamma \, dV = \gamma A \, ds$$
$$I_Y = \int s^2 \sin^2 \theta \, \gamma A \, ds$$
$$I_Y = \gamma A \sin^2 \theta \int_0^l s^2 \, ds$$
$$I_Y = \tfrac{1}{3} \gamma A \sin^2 \theta \, l^3$$

$$M = \gamma A l$$
$$I_Y = \tfrac{1}{3} M l^2 \sin^2 \theta$$

If the axis is normal to the rod, $\theta = 90°$ and $I = \tfrac{1}{3}Ml^2$.

Problems

1. Derive the expressions for the moment of inertia and radius of gyration of a homogeneous right parallelepiped whose sides are a, b, and c, with respect to a geometric axis parallel to side c.

Ans. $I = \dfrac{M}{12}(a^2 + b^2);\ k = \sqrt{\dfrac{a^2 + b^2}{12}}$.

2. Derive the expressions for the moment of inertia and radius of gyration of an elliptic cylinder with major and minor semiaxes a and b, and length l, with respect to its geometric axis.

Ans. $I = \dfrac{M}{4}(a^2 + b^2);\ k = \tfrac{1}{2}\sqrt{a^2 + b^2}$.

3. Derive the expressions for the moment of inertia and radius of gyration of a homogeneous right circular cone with respect to its geometric axis. The radius of the base is r and the altitude is h.

Ans. $I = \tfrac{3}{10}Mr^2;\ k = 0.5477r$.

4. Derive the expressions for the moment of inertia and radius of gyration of a slender rod with respect to a centroidal axis normal to the rod.

Ans. $I = \tfrac{1}{12}Ml^2;\ k = l/3.464$.

5. Derive the expression for the moment of inertia of a hollow right circular cylinder whose outside radius is r_1 and whose inside radius is r_2, with respect to its geometric axis. *Ans.* $I = \tfrac{1}{2}M(r_1^2 + r_2^2)$.

114. The Transfer Formula for Moment of Inertia of Mass.— In case the axis of rotation of a body is not a centroidal axis, the direct computation of $\int \rho^2\, dM$ is usually very difficult. A simpler method is to obtain a relation between the moment of inertia with respect to the axis of rotation and that with respect to a parallel centroidal axis.

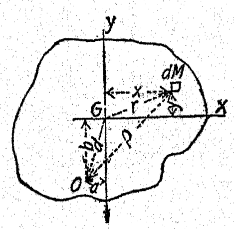


FIG. 341.

Figure 341 represents a section of the body perpendicular to the inertia axis which passes through G. Let O be the point where any parallel axis cuts the section. Then, with respect to the axis through O,

$$I = \int \rho^2\, dM$$

From the figure,

$$\rho^2 = (x + a)^2 + (y + b)^2 = x^2 + y^2 + a^2 + b^2 + 2ax + 2by$$
$$x^2 + y^2 = r^2 \text{ and } a^2 + b^2 = d^2$$

$$I = \int r^2\,dM + d^2\int dM + 2a\int x\,dM + 2b\int y\,dM$$

$$\int r^2\,dM = I_G;\ \int dM = M;\ \int x\,dM = \bar{x}M;\ \int y\,dM = \bar{y}M$$

The quantities $\bar{y}M$ and $\bar{x}M$ both equal zero, since the axes from which x and y are measured are the centroidal axes. Then

$$\mathbf{I = I_G + Md^2}$$

The moment of inertia of a body with respect to any axis is equal to the moment of inertia with respect to a parallel centroidal axis plus the product of the mass of the body and the square of the distance between the axes.

If the equation above is divided by M, it becomes

$$\frac{I}{M} = \frac{I_G}{M} + d^2$$

or

$$k^2 = k_G{}^2 + d^2$$

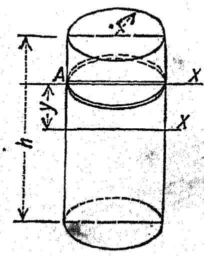


FIG. 342.

EXAMPLE 1

Derive the expression for the moment of inertia of a right circular cylinder whose radius is r and altitude h with respect to a centroidal axis parallel to the base.

Solution.—The cylinder may be divided into circular plates, each of thickness dy, one of which is shown at A in Fig. 342. The moment of inertia of plate A with respect to its own central axis X' is $\frac{1}{4}dMr^2$, as shown in Example 2, Art. 112, and with respect to axis X is $\frac{1}{4}dMr^2 + dMy^2$. Since $dM = \gamma\pi r^2\,dy$, the moment of inertia of all the plates, or the entire cylinder, with respect to axis X is

$$I_X = \frac{1}{4}\gamma\pi r^4\int_{-\frac{h}{2}}^{+\frac{h}{2}} dy + \gamma\pi r^2\int_{-\frac{h}{2}}^{+\frac{h}{2}} y^2\,dy$$

$$= \frac{1}{4}\gamma\pi r^4 h + \frac{1}{3}\gamma\pi r^2\frac{h^3}{4}$$

Since $M = \gamma\pi r^2 h$,

$$I_X = M\left(\frac{r^2}{4} + \frac{h^2}{12}\right)$$

It will be noted that if h becomes negligible compared with r, I_X becomes $\frac{1}{4}Mr^2$. If r becomes negligible compared with h, I_X becomes $\frac{1}{12}Mh^2$, as for a slender rod.

EXAMPLE 2

Derive the expression for the moment of inertia of a right circular cone of height h and radius of base r with respect to an axis through the vertex parallel to the base.

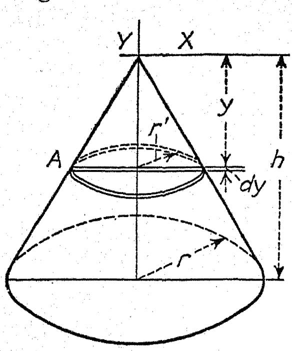


FIG. 343.

Solution.—The cone may be divided into circular plates, each of thickness dy and variable radius r', one of which is shown at A, Fig. 343. The value of I_X for this plate is $\frac{1}{4}dM(r')^2 + dMy^2$. From similar triangles in the section made by the XY plane, $r'/r = y/h$, or $r' = ry/h$. $dM = \gamma\pi(r')^2\,dy = \gamma\pi\frac{r^2}{h^2}y^2\,dy$. For the entire cone,

$$I_X = \frac{1}{4}\gamma\pi\frac{r^4}{h^4}\int_0^h y^4\,dy + \gamma\pi\frac{r^2}{h^2}\int_0^h y^4\,dy$$

$$= \frac{1}{4}\gamma\pi\frac{r^4}{h^4}\frac{h^5}{5} + \gamma\pi\frac{r^2}{h^2}\frac{h^5}{5}$$

Since $M = \frac{1}{3}\gamma\pi r^2 h$,

$$I_X = \frac{3M}{5}\left(\frac{r^2}{4} + h^2\right)$$

Problems

1. Derive the expressions for the moment of inertia and radius of gyration of a right circular cylinder with respect to a diameter of the base.

Ans. $I = M\left(\frac{r^2}{4} + \frac{h^2}{3}\right)$.

2. A rectangular parallelepiped has sides a, b, and c. Derive the expression for its moment of inertia with respect to the central axis in the ac face, parallel to c. *Ans.* $I = M\left(\frac{a^2}{12} + \frac{b^2}{3}\right)$.

3. Derive the expression for the moment of inertia of a sphere with respect to a tangent. *Ans.* $I = \frac{7}{5}Mr^2$.

4. Derive the expression for the moment of inertia of a right circular cone with respect to a diameter of the base. *Ans.* $I = \frac{3}{5}M\left(\frac{r^2}{4} + \frac{h^2}{6}\right)$.

5. Derive the expression for the moment of inertia of a right circular cone with respect to a centroidal axis parallel to the base.

Ans. $I = \frac{3}{5}M\left(\frac{r^2}{4} + \frac{h^2}{16}\right)$.

115. Moment of Inertia of Composite Bodies.—Rotating bodies for which the moment of inertia with respect to the axis of rotation must be computed are often composed of several simple parts. For example, if an ordinary flywheel consisting of a hub, spokes, and rim is keyed to an axle and rotates with it in bearings, the rotating assembly consists of (1) a solid circular

cylinder (axle), (2) a hollow circular cylinder (hub), (3) some slender rods (spokes), and (4) a second hollow circular cylinder (rim).

As stated in Art. 112, no unit is used in connection with the moment of inertia of mass, but it is used as though it were an abstract number. It is necessary, however, that the quantities entering into the computation be given in the proper units—pounds for the weight, feet per second per second for g, and feet for all dimensions.

EXAMPLE

Compute the moment of inertia and radius of gyration of a grindstone 3 ft. in diameter and 4 in. thick with respect to its axis of rotation. Sandstone weighs 150 lb./cu. ft.

Solution.—The grindstone is a cylinder, and its axis of rotation is its geometric axis, for which $I = \frac{1}{2}Mr^2$. The volume of the cylinder is $\pi r^2 h = \pi 1.5^2 \times \frac{1}{3} = 2.36$ cu. ft. The weight $W = 2.36 \times 150 = 354$ lb. The mass $M = 354/32.2 = 11$.

$$I = \frac{1}{2}11 \times 1.5^2 = 12.38$$

$$k^2 = \frac{I}{M} = \frac{12.38}{11} = 1.126$$

$$k = \sqrt{1.126} = 1.06 \text{ ft.}$$

Problems

1. A steel circular saw is 4 ft. in diameter and $\frac{3}{16}$ in. thick. Compute its moment of inertia with respect to its geometric axis. Steel weighs 490 lb./cu. ft. *Ans.* $I = 5.98$.

2. Compute the moment of inertia and the radius of gyration of a steel rod 1 in. square and 4 ft. long with respect to a centroidal axis normal to the rod. Compute the percentage of error in I by considering it as a slender rod instead of a parallelepiped.

Ans. $I = 0.565$; $k = 1.154$ ft.; 0.043 of 1 per cent.

3. Compute the moment of inertia and radius of gyration of the rod described in Prob. 2 with respect to an axis through one end of the rod at an angle of 20° with the rod. *Ans.* $I = 0.2644$; $k = 0.79$ ft.

4. Compute the moment of inertia and radius of gyration of a hollow cast-iron sphere 16 in. outside diameter and 12 in. inside diameter with respect to a diameter. Cast iron weighs 450 lb./cu. ft.

Ans. $I = 2.353$; $k = 0.484$ ft.

5. A cast-iron governor ball is 5 in. in diameter and is connected to the axis of rotation by a steel rod 1 in. in diameter and 20 in. long. Compute the moment of inertia and the radius of gyration of the ball and rod when the rod makes an angle of 40° with the axis of rotation.

Ans. $I = 0.832$; $k = 1.12$ ft.

6. Each side of the hexagonal cast-iron plate shown in Fig. 344 is 8 in., and its thickness is 1 in. Assuming it to be a thin plate, compute I_X and I_Y with respect to the coordinate axes in its central plane. Compute I_Z. Check the value of I_Z by the method of Example 1 of Art. 113.

Ans. $I_X = I_Y = 0.1244$; $I_Z = 0.2488$.

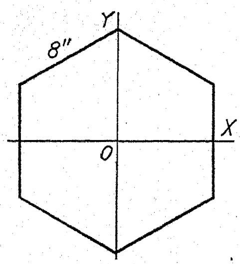


FIG. 344.

7. A cast-iron flywheel has a rim 12 ft. outside diameter, 11 ft. inside diameter, and a face of 2 ft. It has six spokes or arms, elliptic in cross section, with axes 6 and 4 in., respectively. The hub is 18 in. outside diameter, 6 in. inside diameter, and 20 in. long. The wheel is keyed to a steel axle 6 in. in diameter and 20 ft. long. Compute the moment of inertia and the radius of gyration of the wheel and axle with respect to the axis of rotation. Consider the spokes to be slender rods. *Ans.* $I = 17{,}347$; $k = 5.16$ ft.

116. Moment of Inertia by Experiment.—If the form of a body is such that its moment of inertia cannot be computed readily by integration, it may be determined experimentally with a fair degree of accuracy. There are several methods, but the one most readily applicable to problems occurring in engineering is the *pendulum method.* As will be shown in Art. 134, the radius of gyration of a compound pendulum is given by

$$k = \frac{T}{2\pi}\sqrt{gd}$$

in which T is the time of one complete oscillation, g is the acceleration of gravity, and d is the distance from the axis of rotation to the parallel centroidal axis. The axis of rotation must be parallel to the axis for which the moment of inertia is required. If the body is vibrated, and time T of one oscillation determined, k may be computed. Then $I_O = Mk^2$, in which I_O is the moment of inertia with respect to the axis of rotation. The moment of inertia with respect to the parallel centroidal axis is given by

$$I_G = I_O - Md^2$$

From this, if desired, the moment of inertia with respect to any parallel axis may be computed.

Problems

1. A pair of 33-in. cast-iron freight-car wheels and their connecting axle weighed 700 lb. When suspended from knife edges 4 ft. from the axis of

the wheels, they vibrated 100 times (complete oscillations) in 3 min. 43.7 sec. Determine I and k with respect to their centroidal axis.

Ans. $I = 6.99$; $k = 0.568$ ft.

2. The connecting rod of a Corliss engine weighed 267 lb. Its center of gravity was 48.5 in. from the center of the crosshead pin. When suspended by a chain from a point 8 in. above the center of the crosshead pin, it made 20 complete vibrations in 50.9 sec. Compute I_G with respect to the centroidal axis and I_O with respect to the crosshead pin. *Ans.* $I_G = 22.4$; $I_O = 157.4$.

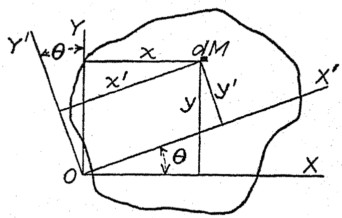


FIG. 345.

117. Moment of Inertia of a Thin Plate with Respect to Inclined Axes. Let Fig. 345 represent any thin plate, and OX and OY any two rectangular axes in the central plane of the plate. Also, let OX' and OY' be a pair of rectangular axes through point O at the angle θ with the original pair.

$$I_{X}' = \int (y')^2\, dM$$

In terms of y, x, and θ, $y' = y \cos \theta - x \sin \theta$, and $(y')^2 = y^2 \cos^2 \theta - 2xy \sin \theta \cos \theta + x^2 \sin^2 \theta$. Then

$$\begin{aligned} I_X' &= \cos^2 \theta \int y^2\, dM + \sin^2 \theta \int x^2\, dM - 2 \sin \theta \cos \theta \int xy\, dM \\ &= I_X \cos^2 \theta + I_Y \sin^2 \theta - 2 \sin \theta \cos \theta \int xy\, dM \end{aligned}$$

In terms of 2θ, since $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$, $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$, and $2 \sin \theta \cos \theta = \sin 2\theta$, the expression above becomes

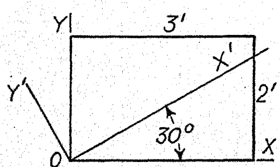


FIG. 346.

$$I_X' = \tfrac{1}{2}(I_X + I_Y) + \tfrac{1}{2}(I_X - I_Y) \cos 2\theta - \sin 2\theta \int xy\, dM$$

Similarly, since $x' = x \cos \theta + y \sin \theta$,

$$I_Y' = \tfrac{1}{2}(I_X + I_Y) - \tfrac{1}{2}(I_X - I_Y) \cos 2\theta + \sin 2\theta \int xy\, dM$$

EXAMPLE

The plate shown in Fig. 346 is cast iron, 3 ft. wide, 2 ft. high, and 1 in. thick. Compute I_X'.

Solution.

$$M = \frac{3 \times 2 \times 450}{12 \times 32.2} = 7$$

$$I_X = \tfrac{1}{3} \times 7 \times 2^2 = 9.33$$
$$I_Y = \tfrac{1}{3} \times 7 \times 3^2 = 21$$
$$\int xy\,dM = \gamma t \int_0^3 \int_0^2 x\,dx\,y\,dy = 9\gamma t = 10.5$$
$$I_{X'} = \frac{30.33}{2} - \frac{11.67}{2} \cos 60° - 10.5 \sin 60°$$
$$I_{X'} = 3.157$$

Problems

1. Compute the moment of inertia of the plate shown in Fig. 346 with respect to the Y' axis. *Ans.* $I_{Y'} = 27.177$.

2. Compute $I_{X'}$ and $I_{Y'}$ for the plate shown in Fig. 346 if $\theta = 15°$. *Ans.* $I_{X'} = 4.865$; $I_{Y'} = 25.469$.

118. Product of Inertia of a Thin Plate.—The expression $\int xy\,dM$ obtained in Art. 117 is called the *product of inertia of mass* and will be denoted by H. The product of inertia is always taken with respect to a pair of rectangular axes. Since either x or y can be negative, and the other coordinate positive, it is evident that H may be either positive or negative.

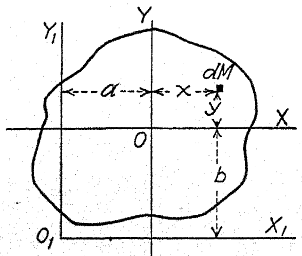


FIG. 347.

If the Y axis is an axis of symmetry of the plate, it is seen that for every term $(+x)y\,dM$ there is a numerically equal negative term $(-x)y\,dM$, and therefore the summation must be zero. Similarly, if the X axis is an axis of symmetry of the plate, it is seen that for every term $x(+y)dM$ there is a numerically equal negative term $x(-y)dM$, and the summation is zero.

If either coordinate axis is an axis of symmetry for a thin plate, the product of inertia with respect to that pair of axes is zero.

119. The Transfer Formula for Product of Inertia.—In Fig. 347, let O be the center of gravity of the thin plate, OX and OY any pair of coordinate axes in the central plane of the plate through the center of gravity, and O_1X_1 and O_1Y_1 any other pair of rectangular axes in the central plane of the plate parallel to OX and OY respectively. Let the distance between O_1Y_1 and OY be a, and the distance between O_1X_1 and OX be b. With respect to the axes O_1X_1 and O_1Y_1 the product of inertia is

$$H_{O_1} = \int (a + x)(b + y)dM$$
$$= \int ab\, dM + \int ay\, dM + \int bx\, dM + \int xy\, dM$$
$$= abM + a\bar{y}M + b\bar{x}M + H_O$$

The second and third terms become zero, since $\bar{y} = 0$ and $\bar{x} = 0$. H_O is the product of inertia with respect to axes OX and OY.

$$H_{O_1} = H_O + abM$$

The quantities a and b may be either positive or negative, so the term abM may be either positive or negative. If the center of gravity of the plate is in the first or third quadrant of the axes O_1X_1, O_1Y_1, the term abM is positive. If it is in the second or fourth quadrant, the term abM is negative.

If a plate is composed of several simple parts, its product of inertia with respect to any coordinate pair of axes is the algebraic sum of the products of inertia of the several parts with respect to the same axes.

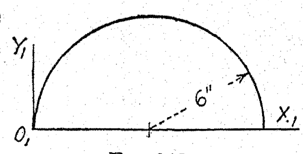


FIG. 348.

EXAMPLE

Compute the product of inertia of the semicircular plate shown in Fig. 348 with respect to the X_1 and Y_1 axes. The plate is aluminum (165 lb./cu. ft.), with radius of 6 in. and a thickness of ½ in.

Solution.—The Y axis through the center of gravity is an axis of symmetry, so $H_O = 0$, and $H_{O_1} = abM$.

$$M = \frac{\pi \times 165}{8 \times 24 \times 32.2} = 0.0838$$
$$H_{O_1} = \frac{4 \times 0.0838}{2 \times 2 \times 3\pi} = +0.0089$$

Problems

1. Compute the product of inertia of a rectangular steel plate 3 ft. wide, 4 ft. high, and ¾ in. thick with respect to coordinate axes in the central plane of the plate, parallel to the edges of the plate through the lower right-hand corner. *Ans.* -34.26.

2. Compute the product of inertia of a circular cast-iron plate 3 ft. in diameter and 2 in. thick with respect to coordinate axes in the central plane of the plate, tangent to the edges of the plate. *Ans.* ± 37.04.

3. The plate shown in Fig. 349 is steel and is 3 in. thick. Compute H_{O_1}. *Ans.* $+45.6$.

4. Compute H_{O_1} for the left half of the triangular plate shown in Fig. 349. (It is simpler to integrate directly.) *Ans.* $+17.1$.

5. The plate shown in Fig. 350 is steel, 1 in. thick. Compute H_{O_1} for axes in the central plane of the plate. *Ans.* $+0.4755$.

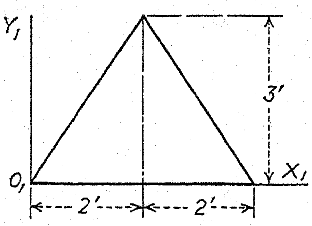


FIG. 349.

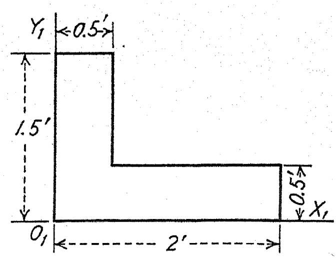


FIG. 350

120. Principal Axes of Inertia of a Thin Plate.—As derived in Art. 117, the value of the moment of inertia of a thin plate with respect to an axis at an angle θ with some original axis is given by the expression

$$I_X' = \tfrac{1}{2}(I_X + I_Y) + \tfrac{1}{2}(I_X - I_Y)\cos 2\theta - H_O \sin 2\theta.$$

As the value of the angle θ varies, the value of I_X' varies. The values of the angle θ to give maximum and minimum values of I_X' may be obtained by differentiating the expression for I_X' with respect to θ and equating to zero.

$$\frac{dI_X'}{d\theta} = (I_Y - I_X)\sin 2\theta - 2H_O \cos 2\theta$$

For maximum and minimum values of I_X',

$$(I_Y - I_X)\sin 2\theta - 2H_O\cos 2\theta = 0$$

$$\tan 2\theta = \frac{2H_O}{I_Y - I_X}$$

Two values of 2θ differing by 180° will be obtained from this equation, and therefore two values of θ differing by 90°. One value is the angle for the maximum I_X' and the other is the value for the minimum I_X'. The maximum and minimum moments of inertia are called the *principal moments of inertia,* and the corresponding axes are called the *principal axes.*

If either the X or Y axis is an axis of symmetry, $H_O = 0$ by Art. 118, and $\tan 2\theta = 0$. $2\theta = 0°$, or 180°. $\theta = 0°$, or 90°, so the X and Y axes are the principal axes.

EXAMPLE

Locate the principal axes through point O_1 for the plate shown in Fig. 348. Compute the maximum and minimum moments of inertia.

Solution.—From the result of the example in Art. 119,

$$2H_{O_1} = 0.0178$$
$$I_{Y_1} = 0.02619$$
$$I_{X_1} = 0.00524$$
$$\tan 2\theta = \frac{0.0178}{0.02095} = 0.85$$
$$2\theta = 40°20' \text{ or } 220°20'$$
$$\theta = 20°10' \text{ or } 110°10'$$

$$\text{Min. } I_X' = \tfrac{1}{2}(0.02619 + 0.00524) - \tfrac{1}{2}(0.02619 - 0.00524) \times 0.76 - 0.0089 \times 0.648$$

$$\text{Min. } I_X' = 0.00201$$
$$\text{Max. } I_X' \text{ (or } I_Y') = 0.02943$$

Problems

1. Locate the principal axes through point O for the rectangular plate shown in Fig. 346. Compute the maximum and minimum moments of inertia. *Ans.* $\theta = 30°28'$ or $120°28'$; $I_Y' = 27.18$; $I_X' = 3.16$.

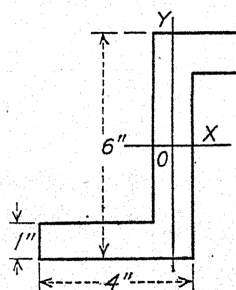


FIG. 351.

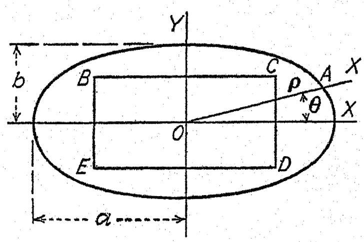


FIG. 352.

2. Locate the center of gravity of the plate shown in Fig. 350. Locate the principal axes through this point. Compute the maximum and minimum moments of inertia.

Ans. $\bar{x} = \frac{3}{4}$ ft.; $\bar{y} = \frac{1}{2}$ ft.; $\theta = 63°25'$ or $153°25'$; max. $I = 0.7985$; min. $I = 0.1921$.

3. The Z-shaped zinc plate shown in Fig. 351 is $\frac{1}{2}$ in. thick. Locate the principal axes through point O. Compute the maximum and minimum moments of inertia for axes in the central plane of the plate. The weight of zinc is 440 lb./cu. ft.

Ans. $\theta = 57°10'$ or $147°10'$; max. $I = 0.002077$; min. $I = 0.000259$.

121. Ellipse of Inertia of a Thin Plate.—If the principal axes of a thin plate, such as $BCDE$, Fig. 352, are made coincident with the X and Y axes, the angle between them is zero, and therefore

the quantity $\frac{2H_O}{I_Y - I_X}$ must equal zero. It is thus seen that for the principal axes the product of inertia $H_O = 0$, and the expression for I_X' with respect to any axis X' at an angle θ with the principal X axis becomes

$$I_X' = I_X \cos^2 \theta + I_Y \sin^2 \theta$$

The variation in the value of I_X' may be shown graphically by laying off along axis OX' a distance $\rho = OA$ which is inversely proportional to the square root of I_X'. Let K be any convenient constant. Then

$$\rho = \frac{K}{\sqrt{I_X'}}$$

$$I_X' = \frac{K^2}{\rho^2} = I_X \cos^2 \theta + I_Y \sin^2 \theta$$

$$K^2 = I_X \rho^2 \cos^2 \theta + I_Y \rho^2 \sin^2 \theta$$

In this equation, $\rho \cos \theta = x$ of point A and $\rho \sin \theta = y$ of point A.

$$K^2 = I_X x^2 + I_Y y^2$$

$$\frac{x^2}{(K^2/I_X)} + \frac{y^2}{(K^2/I_Y)} = 1$$

This is the equation of an ellipse, with its major semiaxis $a = K/\sqrt{I_X}$, and its minor semiaxis $b = K/\sqrt{I_Y}$.

EXAMPLE

In Fig. 352, let the plate be steel, with the dimension $BC = 4$ ft., $CD = 2$ ft., and the thickness ⅜ in. Using $K = 4$, compute the value of ρ for $\theta = 0°, 15°, 30°, 45°, 60°, 75°$, and $90°$.

Solution.

$$M = \frac{2 \times 4 \times 3 \times 490}{32.2 \times 8 \times 12} = 3.81$$

$$I_X = \frac{3.81 \times 4}{12} = 1.27$$

$$I_Y = \frac{3.81 \times 16}{12} = 5.08$$

At $\theta = 0°$,

$$I_X' = I_X = 1.27; \sqrt{1.27} = 1.127; \rho = \frac{4}{1.127} = 3.56 \text{ ft.}$$

At $\theta = 15°$,

$$I_X' = 1.27 \times 0.966^2 + 5.08 \times 0.259^2 = 1.523; \sqrt{1.523} = 1.235;$$

$$\rho = \frac{4}{1.235} = 3.24 \text{ ft.}$$

At $\theta = 30°$,

$$I_X' = 1.27 \times 0.866^2 + 5.08 \times 0.5^2 = 2.222; \sqrt{2.222} = 1.49;$$

$$\rho = \frac{4}{1.49} = 2.68 \text{ ft.}$$

At $\theta = 45°$,

$$I_X' = 1.27 \times 0.707^2 + 5.08 \times 0.707^2 = 3.175; \sqrt{3.175} = 1.78;$$

$$\rho = \frac{4}{1.78} = 2.245 \text{ ft.}$$

At $\theta = 60°$,

$$I_X' = 1.27 \times 0.5^2 + 5.08 \times 0.866^2 = 4.128; \sqrt{4.128} = 2.035;$$

$$\rho = \frac{4}{2.035} = 1.97 \text{ ft.}$$

At $\theta = 75°$,

$$I_X' = 1.27 \times 0.259^2 + 5.08 \times 0.966^2 = 4.825; \sqrt{4.825} = 2.20;$$

$$\rho = \frac{4}{2.20} = 1.82 \text{ ft.}$$

At $\theta = 90°$,

$$I_X' = I_Y = 5.08; \sqrt{5.08} = 2.254; \rho = \frac{4}{2.254} = 1.78 \text{ ft.}$$

Problems

1. In Fig. 352, let the plate be cast iron, with the dimension $BC = 3$ ft., $CD = 2$ ft., and the thickness 1 in. With $K = 5$, compute the value of ρ for $\theta = 0°$, 15°, 30°, 60°, and 90°.

Ans. 3.28 ft.; 3.14 ft.; 2.85 ft.; 2.35 ft.; 2.18 ft.

2. With $K = 0.05$, compute the value of ρ for the minimum principal axis of the plate shown in Fig. 351. Compute also the value of ρ for $\theta = 15°$, 32°50′ (the Y axis), 60°, and 90°.

Ans. 3.115 ft.; 2.56 ft.; 1.77 ft.; 1.24 ft.; 1.095 ft.

GENERAL PROBLEMS ON MOMENT OF INERTIA OF MASSES

1. For what ratio of the diameter to the length is there an error of 1 per cent in the computation of I of a solid circular cylinder with respect to a diameter of one end if the cylinder is considered as a slender rod?

Ans. 1:4.31.

2. Derive the expressions for the moment of inertia of a right elliptic cylinder with respect to the major and minor axes of the centroidal cross section.

Ans. $I_X = M\left(\frac{b^2}{4} + \frac{h^2}{12}\right)$; $I_Y = M\left(\frac{a^2}{4} + \frac{h^2}{12}\right)$.

3. Derive the expressions for the moment of inertia of a right elliptic cylinder with respect to the major and minor axes of one end of the cylinder.

Ans. $I_X = M\left(\frac{b^2}{4} + \frac{h^2}{3}\right)$; $I_Y = M\left(\frac{a^2}{4} + \frac{h^2}{3}\right)$.

4. A cast-iron governor ball 4.5 in. in diameter has for its arm a steel rod 20 in. long and 1 in. in diameter. Compute the moment of inertia of the

ball and rod when they are rotating about a vertical axis through the free end of the rod at an angle of 20° with the axis of rotation.

Ans. $I = 0.1762$.

5. Compute the moment of inertia of the governor ball and rod described in Prob. 4 when they are at an angle of 60° with the axis of rotation.

Ans. $I = 1.096$.

6. A steel disk 9 in. in diameter and 3 in. thick is rotating about an axis 3 in. from its geometric axis. Compute its moment of inertia with respect to the axis of rotation. *Ans.* $I = 0.223$.

7. A rotating bowl consists of a cast-iron hemisphere with a cone cut out, as shown in cross section in Fig. 353. Compute the moment of inertia of the bowl with respect to the Y axis. *Ans.* $I = 2.942$.

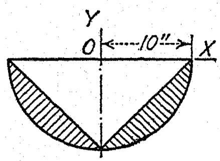


FIG. 353.

8. Compute the moment of inertia of a hollow cast-iron sphere, 8 in. outside diameter and 6 in. inside diameter, with respect to an axis 1 ft. from its center.

Ans. $I = 1.328$.

9. Compute the moment of inertia of a hollow steel cylinder 4 ft. outside diameter, 3 ft. inside diameter, and 16 in. long with respect to an element of the inside cylindrical surface. *Ans.* $I = 598$.

10. Compute the moment of inertia of the hollow cylinder described in Prob. 9 with respect to an element of the outside cylindrical surface.

Ans. $I = 792$.

11. A steel disk 30 in. in diameter and 3 in. thick has a cylindrical hole 6 in. in diameter at the center and another 4 in. in diameter with its center 10 in. from the center of the disk. Locate its center of gravity. Compute its moment of inertia with respect to the centroidal axis parallel to the geometric axis. *Ans.* 0.0157 ft.; $I = 14.326$.

12. A flywheel governor consists of a cast-iron plate 4 in. wide, 22 in. long, and 1 in. thick (Fig. 354), to which are fastened the cast-iron cylinders

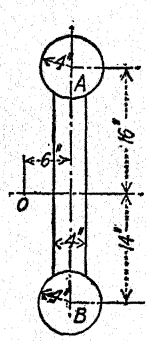


FIG. 354.

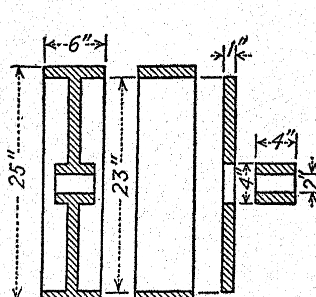


FIG. 355.

A and B. Cylinder A is 8 in. in diameter and 3 in. thick, and cylinder B is 8 in. in diameter and 5 in. thick. Compute the moment of inertia of the governor with respect to an axis through O parallel to the axes of the cylinders. *Ans.* $I_O = 6.323$.

13. A cast-iron pulley with a solid web has dimensions as shown in Fig. 355. Compute its moment of inertia with respect to its geometric axis.

Ans. $I = 5.214$.

14. The cast-iron flywheel shown in Fig. 356 has six spokes, elliptic in cross section, which may be considered as slender rods. Compute the moment of inertia of the wheel with respect to its geometric axis.

Ans. $I = 107.60$.

15. Compute the moment of inertia of the flywheel described in Prob. 14 by the common method of approximation, as follows: Neglect the hub. Consider the wheel to have three double spokes, each 51 in. long.

Ans. $I = 107.46$.

16. A steel plate is 2 ft. square and 1 in. thick. Compute its moment of inertia with respect to an axis in its central plane passing through the middle of one side and the opposite corner, as shown in Fig. 357.

Ans. $I_{X'} = 2.704$.

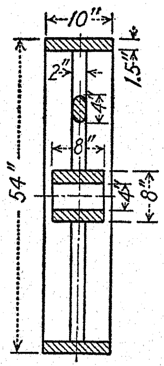


FIG. 356.

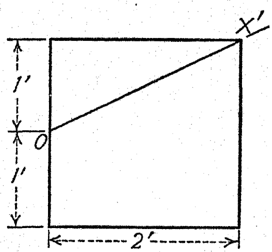


FIG. 357.

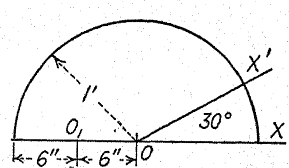


FIG. 358

17. In Fig. 357, let the horizontal dimension be changed to 4 ft., all other dimensions remaining the same. Compute the moment of inertia of the plate with respect to axis OX' passing through the middle of the left edge and the upper right-hand corner. *Ans.* $I_{X'} = 6.36$.

18. The thin plate shown in Fig. 358 weighs 8.2 lb. Compute its moment of inertia with respect to axis OX'. *Ans.* $I_{X'} = 0.0635$.

19. In Fig. 358, solve for the maximum and minimum moments of inertia of the plate with respect to axes through point O'.

Ans. Max. $I = 0.1582$; min. $I = 0.0328$.

20. Prove that for any thin plate with equal edges, the moment of inertia with respect to any centroidal axis in the central plane of the plate is a constant.

CHAPTER XIV

ROTATION OF RIGID BODIES

122. Angular Displacement.—If a particle describes a plane curvilinear motion with a constant radius r, the motion is called *rotation*, and the angle described by the radius is called *angular displacement.* The unit of angular displacement is the *radian.* The radian is the angle at the center subtended by an arc equal in length to the radius. In Fig. 359, the length of the arc AB is equal to the radius r, so the angle AOB is one radian. Let the length of the arc ABC be s. Since any angle is proportional to its subtending arc,

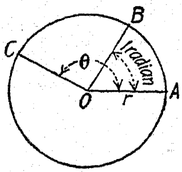


Fig. 359.

$$\theta = \frac{s}{r}$$

or

$$\mathbf{s = r\theta}$$

There are 2π radians subtended by a complete circumference. Hence, 2π radians $= 360°$, and 1 radian $= 360°/2\pi = 57°.3$. (More accurately, $57°.29578$.)

Angular displacement in the counterclockwise direction is usually considered to be positive; that in the clockwise direction, negative.

If a rigid body has a motion of rotation, all particles along the axis of rotation remain fixed in position, while all other particles of the body describe circular paths about the fixed axis. It will be noted that any line in the plane of motion of a rotating rigid body has the same angular displacement as that of the radius of any point in the body.

Problems

1. Reduce to radians: 30°; 120°; 400°; 3.6 revolutions.

Ans. 0.5236 rad.; 2.0944 rad.; 6.981 rad.; 22.621 rad.

2. Reduce to degrees: 1.6 rad.; 3π rad.; 6.4 rev.

Ans. 91°41′; 540°; 2304°.

3. Reduce to revolutions: 75°; 420°; 6 rad.; 22.3 rad.

Ans. 0.2083 rev.; 1.167 rev.; 0.955 rev.; 3.55 rev.

4. If the diameter of a circle is 3 ft., what is the length of an arc of 75°; of 2.5 radians? *Ans.* 1.96 ft.; 3.75 ft.

123. Angular Velocity.—Angular velocity is the time rate of the angular displacement of the radius of any rotating particle or of any line in the plane of rotation of a rigid body. If equal angular displacements occur in equal intervals of time, the angular velocity is constant. Let ω represent the angular velocity in radians per second. Then if θ represents the angular displacement in time t, the rate of angular displacement, or angular velocity, is given by

$$\omega = \frac{\theta}{t}$$

If the angular velocity varies, the *average* angular velocity for any small interval of time Δt is given by

$$\omega = \frac{\Delta\theta}{\Delta t}$$

The instantaneous angular velocity is given by

$$\omega = \frac{d\theta}{dt}$$

By differentiating the expression $s = r\theta$ with respect to t, the relation between the linear and angular velocities is obtained.

$$\frac{ds}{dt} = r\frac{d\theta}{dt}$$

$$\mathbf{v} = \mathbf{r}\boldsymbol{\omega}$$

Angular velocity is commonly given in revolutions per minute (r.p.m.), which must usually be reduced to radians per second for the solutions of problems.

$$1 \text{ r.p.m.} = \frac{1}{60} \text{ rev./sec.} = \frac{2\pi}{60} \text{ rad./sec.}$$

Angular velocity has sign, counterclockwise velocity being usually taken as positive, and clockwise velocity as negative.

Since angular velocity involves only magnitude and direction, it is a vector quantity and may be represented graphically by a

vector. In the same way as for vectors of couples, the vector is drawn parallel to the axis of rotation with the arrow pointing in the direction from which the rotation appears counterclockwise, or positive. Vectors representing angular velocity may be combined graphically into their resultant vector which represents the resultant velocity. Conversely, the vector representing an angular velocity may be resolved into component vectors.

Problems

1. A pulley 16 in. in diameter rotates at a speed of 1800 r.p.m. Compute its angular velocity in radians per second and the tangential velocity of the rim in feet per second. *Ans.* $\omega = 188.5$ rad./sec.; $v = 125.7$ ft./sec.

2. The smaller of two friction wheels is 4.5 in. in diameter, and the larger is 30 in. in diameter. If the smaller wheel is rotating at 20 r.p.m., compute the angular velocity of the larger wheel and the linear velocity of the rim. *Ans.* $\omega = 0.314$ rad./sec.; $v = 0.3927$ ft./sec.

3. If the linear speed of a belt on a pulley 4 ft. in diameter is 1 mile per minute, what is the angular velocity ω of the pulley? *Ans.* $\omega = 44$ rad./sec.

124. Angular Acceleration.—Angular acceleration is the time rate of change of the angular velocity of the radius of any rotating particle or of any line in the plane of rotation of a rigid body. If the angular velocity is constant, the angular acceleration is zero. If the angular velocity changes by equal amounts in equal time intervals, the angular acceleration is constant. If the angular velocity changes by unequal amounts in equal time intervals, the angular acceleration is variable.

If the angular acceleration is constant, its value may be obtained by dividing the total change in angular velocity by the time t in which the change was made. If α represents the angular acceleration and ω the change in angular velocity,

$$\alpha = \frac{\omega}{t}$$

If the angular acceleration is variable, its instantaneous value at any point is given by

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

By eliminating dt from the two equations $\omega = d\theta/dt$ and $\alpha = d\omega/dt$, the equation

$$\boldsymbol{\omega\, d\omega = \alpha\, d\theta}$$

is obtained.

By differentiating the expression $v = r\omega$ with respect to t, the relation between the tangential and angular accelerations is obtained.

$$\frac{dv}{dt} = \frac{r\,d\omega}{dt}$$

$$\mathbf{a}_t = \mathbf{r}\boldsymbol{\alpha}$$

By Art. 105, $a_n = v^2/r$. Since $v = r\omega$,

$$\mathbf{a}_n = \mathbf{r}\boldsymbol{\omega}^2$$

It is seen that the tangential acceleration varies directly with the radius and with the angular acceleration. The normal acceleration varies with the radius and with the square of the angular velocity. It is independent of the angular acceleration.

The unit of angular acceleration is the radian per second per second. Like angular velocity, angular acceleration is a vector quantity and may be represented graphically by a vector. Angular acceleration has sign, counterclockwise acceleration being usually taken as positive, and clockwise as negative.

Problems

1. What is the normal acceleration of a point on the rim of the pulley referred to in Prob. 3, Art. 123? The wheel was brought up to speed in 20 sec. Compute the average angular acceleration.

Ans. $a_n = 3872$ ft./sec.^{2}; $a_t = 2.2$ ft./sec.^{2}.

2. If the friction wheels described in Prob. 2, Art. 123, were brought from rest up to their running speed in 0.2 sec. with constant acceleration, compute the angular acceleration of each wheel and the tangential acceleration of the point of contact.

Ans. $\alpha_1 = 10.47$ rad./sec.^{2}; $\alpha_2 = 1.57$ rad./sec.^{2}; $a_t = 1.96$ ft./sec.^{2}.

125. Uniform Motion in a Circle.—Let A, Fig. 360, be any particle of a rigid body that has a motion of rotation about axis O with a constant angular velocity of ω radians per second. Since ω is constant, the angular acceleration α and the tangential acceleration a_t are both zero. The normal acceleration is $a_n = r\omega^2$. If M is the mass of the particle, the normal force upon the particle directed toward O must be Ma, or $Mr\omega^2$.

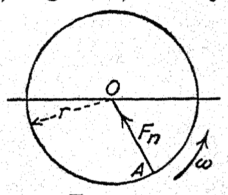


FIG. 360.

The time required for the particle to make one complete revolution is called its *period* and is obtained by dividing the

number of radians in one revolution by the number of radians rotated by the particle per second.

$$T = \frac{2\pi}{\omega}$$

Problems

1. A steam turbine 16 in. in diameter is rotating at 12,500 r.p.m. Compute the period T, the angular velocity ω, and the normal acceleration a_n of a point on the rim.

Ans. $T = 0.0048$ sec.; $\omega = 1309$ rad./sec.; $a_n = 1{,}140{,}000$ ft./sec.^{2}.

2. A body weighing 2 lb. rotates on a smooth horizontal surface at the end of a cord 1.6 ft. long at a speed of 6 rev./sec. Compute the tension in the cord. *Ans.* 141.4 lb.

126. Simple Harmonic Motion.—*Simple harmonic motion* is a rectilinear vibratory motion in which the acceleration is proportional to the displacement from a certain point and is oppositely directed. If a represents the acceleration, s the displacement from the given point, and K a constant,

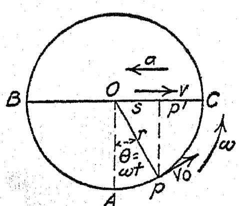


FIG. 361.

$$a = -Ks$$

Since simple harmonic motion is a rectilinear motion, it should logically have been discussed in Chap. XI. It is simpler, however, to develop the equations of this motion by means of the *auxiliary circle*, the circle of rotation that has a radius equal to the amplitude of the motion.

Let Fig. 361 represent the circle of rotation of radius r around which a particle P moves with a uniform angular velocity ω in the positive direction, and let point A be the position of the particle at the instant from which time is measured. The motion of point P', the projection of point P on the diameter BC, is then simple harmonic motion, as will be shown.

The angle of displacement AOP is

$$\theta = \omega t$$

The displacement of P' from O is

$$s = r \sin \omega t$$

The velocity of P' is

$$v = \frac{ds}{dt} = r\omega \cos \omega t$$

This is the horizontal component of the tangential velocity v_0 of point P, as shown in Fig. 362 (a). The acceleration of P' is

$$a = \frac{dv}{dt} = -r\omega^2 \sin \omega t$$

This is the horizontal component of the acceleration of point P, as shown in Fig. 362 (b). Since $r \sin \omega t = s$,

$$a = -\omega^2 s$$

Since the angular velocity ω is constant, ω^2 is also a constant and is equal to the constant K as given above. This shows that the motion of point P' is simple harmonic motion.

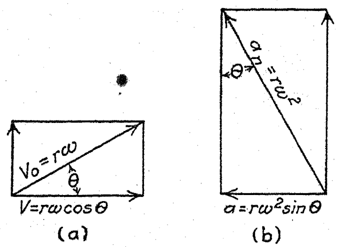


FIG. 362.

The time t required for the motion from O to P' may be obtained from the equation

$$s = r \sin \omega t$$

$$t = \frac{1}{\omega} \sin^{-1} \frac{s}{r}$$

The time T of one complete vibration of P' is the same as that of one complete rotation of particle P, and, as given in Art. 125, is

$$T = \frac{2\pi}{\omega}$$

The piston and slotted slider shown in Fig. 363 have simple harmonic motion if the crankpin P moves around the crankpin

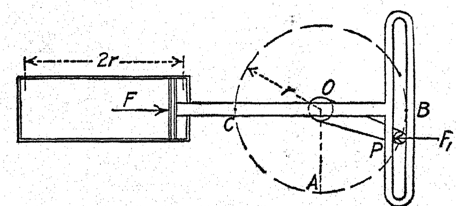


FIG. 363.

circle at a uniform speed. A loaded coil spring, set in vibration axially, moves with simple harmonic motion. If such a coil spring is held vertically and loaded with a weight W to cause a

static elongation e, the scale of the spring in pounds per foot is $C' = W/e$.

$$e = \frac{W}{C'}$$

If the weight W is displaced vertically and then released, it will vibrate up and down past its static position with simple harmonic motion for which the period of vibration is

$$T = \frac{2\pi}{\omega}$$

The period is independent of the amplitude of vibration; and if $s = -1$, $a = \omega^2$, and the unbalanced force on the weight W is

$$C' = Ma = \frac{W}{g}\omega^2$$

$$\omega^2 = g\frac{C'}{W} = \frac{g}{e}$$

$$\omega = \sqrt{\frac{g}{e}}$$

$$T = 2\pi\sqrt{\frac{e}{g}}$$

The motion of the piston and crosshead of an ordinary reciprocating steam engine approximates simple harmonic motion, as does also the motion of a vibrating leaf spring.

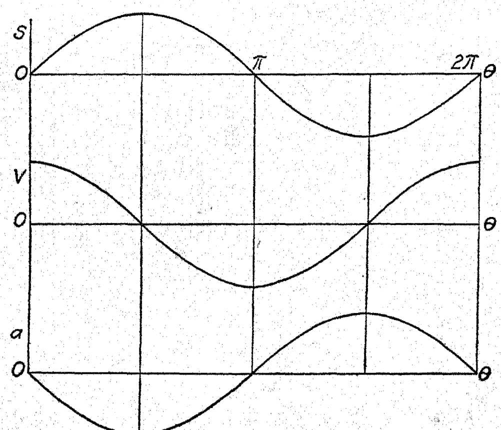


FIG. 364.

Figure 364 shows graphically the manner in which s, v, and a vary with θ. The curves are of course the sine and cosine curves.

EXAMPLE 1

In the mechanism shown in Fig. 363, let $r = 1$ ft., and let the crankpin be rotating at 120 r.p.m. Determine ω, T, v_0, and the maximum acceleration of the piston. If the weight of the piston and slider is 200 lb., and the steam pressure F is zero, what is the pressure F_1 of the crankpin on the slider when the piston is at its end position?

Solution.

$$120 \text{ r.p.m.} = 2 \text{ rev./sec.} = 4\pi \text{ rad./sec.}$$

$$\omega = 4\pi = 12.57 \text{ rad./sec.}$$

$$T = \frac{2\pi}{\omega} = \frac{1}{2} \text{ sec.}$$

$$v_0 = r\omega = 12.57 \text{ ft./sec.}$$

Max. $$a = -\omega^2 r = -158 \text{ ft./sec.}^2$$

At the end position, the only force acting upon the piston and slider in the direction of its motion is the pressure of the crankpin F_1. The acceleration is toward the left, so the force F_1 must act toward the left. From $F = Ma$,

$$F_1 = \frac{200}{32.2} \times 158 = 981 \text{ lb.}$$

EXAMPLE 2

Figure 365 represents a 10-lb. ball at O placed between two horizontal coil springs ~~and~~ supported by a smooth plane. The ball is attached to both springs so that when it is displaced longitudinally, one spring is compressed and the other is elongated. When the ball is at its middle position, neither spring is acting. Each spring has a scale of 25 lb./in. If the ball is displaced 3 in. and then released, compute the velocity v_0 and the period of vibration T. Find the position, velocity, and acceleration of the ball 0.45 sec. after release.

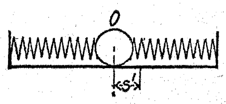


FIG. 365.

Solution.—The force exerted by each spring is twenty-five times the displacement in inches, so the total force exerted by the two springs when the deformation is 3 in. is

$$F = 2 \times 3 \times 25 = 150 \text{ lb.}$$

The numerical value of ω^2 is given by the equation

$$F = Ma = Mr\omega^2$$

$$150 = \frac{10}{32.2 \times 4}\omega^2$$

$$\omega^2 = 1932$$

$$\omega = 43.95 \text{ rad./sec.}$$

$$v_0 = r\omega = 0.25 \times 43.95 = 10.99 \text{ ft./sec.}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{43.95} = 0.14296 \text{ sec.}$$

In 0.45 sec. the ball has made three complete vibrations and has moved $0.45 - 3 \times 0.14296 = 0.02112$ sec. on the fourth vibration. The value

of the angle from the end position is $43.95 \times 0.02112 = 0.9282$ radians. Since ωt is measured from the mean position 90° earlier, $\pi/2$ must be added.

$$\omega t = 0.9282 + 1.5708$$
$$\omega t = 2.499 \text{ rad.} = 143°12'$$

The displacement 0.45 sec. after release is given by the equation

$$s = r \sin 143°12' = 0.15 \text{ ft.}$$

The velocity of the ball at this point is given by the equation

$$v = r\omega \cos 143°12' = -8.79 \text{ ft./sec.}$$

The acceleration of the ball is given by the equation

$$a = -\omega^2 s = -289.8 \text{ ft./sec.}^2$$

Instead of using ωt, the smaller angle that the radius OP makes with either of the axes may be used. In Fig. 366, this angle is $\beta = 36°48'$. Displacement $s = OP' = 0.25 \sin 36°48' = 0.15$ ft.

Velocity $v = -v_0 \cos \beta = -10.99 \times 0.8 = -8.79$ ft./sec.

Normal acceleration $a_n = r\omega^2 = 0.25 \times 1932 = 483$ ft./sec.²

Acceleration $a = -a_n \sin \beta = -483 \times 0.599 = -289.8$ ft./sec.²

Problems

1. In Fig. 363, let the angle POA of the crank with the vertical radius OA be 75°, and let all other data be the same as in the foregoing example. Determine the velocity and the acceleration of the piston. Determine the time since it was in its middle position. If the steam pressure is 300 lb., what is the crankpin pressure F_1?

Ans. 3.25 ft./sec.; 152.63 ft./sec.²; 5/48 sec.; 1246 lb.

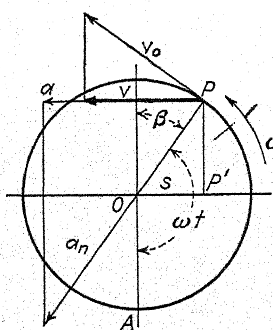


FIG. 366.

2. If the ball in the apparatus shown in Fig. 365 is not attached to the springs, what will be its period of vibration? If the ball is displaced 4 in. to the left and then released, what will be its displacement, velocity, and acceleration 1.1 sec. after release?

Ans. $T = 0.2021$ sec.; $s = 0.31$ ft.; $v = 3.74$ ft./sec.; $a = -300$ ft./sec.².

3. A weight of 50 lb. is hung from a 10-lb. spring. Compute its period of vibration. If the weight is pulled down a further distance of 6 in. and then suddenly released, compute its displacement from its static position, its velocity, and its acceleration after 0.6 sec.

Ans. $T = 0.715$ sec.; $s = -0.265$ ft.; $v = -3.71$ ft./sec.; $a = 20.52$ ft./sec.².

127. Constant Angular Acceleration.

—By definition,

$$\alpha = \frac{d\omega}{dt}$$
$$d\omega = \alpha\, dt$$

For the case in which the acceleration is constant, if ω_0 is the initial velocity and ω the velocity after time t,

$$\int_{\omega_0}^{\omega} d\omega = \alpha \int_0^t dt$$
$$\omega - \omega_0 = \alpha t$$
$$\boldsymbol{\omega = \omega_0 + \alpha t}$$

By definition again,

$$\omega = \frac{d\theta}{dt}$$
$$d\theta = \omega\, dt$$
$$d\theta = \omega_0\, dt + \alpha t\, dt$$
$$\int_0^{\theta} d\theta = \omega_0 \int_0^t dt + \alpha \int_0^t t\, dt$$
$$\boldsymbol{\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2}$$

From Art. 124,

$$\omega\, d\omega = \alpha\, d\theta$$
$$\int_{\omega_0}^{\omega} \omega\, d\omega = \alpha \int_0^{\theta} d\theta$$
$$\frac{\omega^2}{2} - \frac{{\omega_0}^2}{2} = \alpha\theta$$
$$\boldsymbol{\omega^2 = {\omega_0}^2 + 2\alpha\theta}$$

Problems

1. A flywheel is brought from rest up to a speed of 1600 r.p.m. in 1 min. 20 sec. Compute the average angular acceleration α and the number of revolutions required. Compute the velocity at the end of 15 sec.

Ans. 2.094 rad./sec.^{2}; 1067 rev.; 31.41 rad./sec.

2. If the flywheel of Prob. 1 is 18 in. in diameter, compute the tangential acceleration of a point on the rim. Compute the tangential velocity and the normal acceleration of a point on the rim at the instant when full speed is attained.

Ans. a_t = 1.571 ft./sec.^{2}; v = 125.67 ft./sec.; a_n = 21,056 ft./sec.^{2}.

3. A motor rotating at 8000 r.p.m. comes to rest under the action of friction in 64 min. 20 sec. Compute the angular acceleration and the total number of revolutions. *Ans.* −0.217 rad./sec.^{2}; 257,333 rev.

4. The rim of a 33-in. wheel on a brake-shoe testing machine has a speed of 60 m.p.h. when the brake is dropped. It comes to rest when the rim has traveled a tangential distance of 416 ft. Compute the angular acceleration and the number of revolutions. *Ans.* −6.75 rad./sec.^{2}; 48.1 rev.

128. Variable Angular Acceleration.—If in a circular motion the angular acceleration α is variable, its law of variation must be known in order to obtain the equations of motion. The motion of the torsion pendulum will be used as an illustration. The

torsion pendulum consists of a relatively heavy body fixed rigidly to a slender elastic rod by which it is hung in a vertical position. The supporting rod is also firmly fixed to a solid support at its upper end. Within the elastic limit of the material of the rod, the angular displacement θ is proportional to the moment of the displacing couple. If the displacing couple is suddenly removed, the angular acceleration of the pendulum is proportional to the displacement and in the opposite direction.

$$\alpha = -K\theta,$$

K being a constant. Let the angular velocity at the mid-position be ω_0. Since $\omega\,d\omega = \alpha\,d\theta$,

$$\int_{\omega_0}^{\omega} \omega\,d\omega = -K\int_0^{\theta} \theta\,d\theta$$

$$\omega = \sqrt{\omega_0^2 - K\theta^2}$$

Let θ_1 be the maximum value of the angle θ. When $\theta = \theta_1$, $\omega = 0$.

$$0 = \sqrt{\omega_0^2 - K\theta_1^2}$$

$$\omega_0 = \sqrt{K}\,\theta_1$$

Since $\omega = d\theta/dt$, the equation for ω becomes,

$$dt = \frac{d\theta}{\sqrt{\omega_0^2 - K\theta^2}}$$

If time is measured from the instant the body is in the mid-position moving positively, the limits of t are 0 and t, and of θ are 0 and θ.

$$\int_0^t dt = \int_0^{\theta} \frac{d\theta}{\sqrt{\omega_0^2 - K\theta^2}}$$

$$t = \frac{1}{\sqrt{K}} \sin^{-1} \frac{\sqrt{K}}{\omega_0}\theta$$

The time from the mid-position till the pendulum is at rest at its maximum displacement θ_1 is

$$t_1 = \frac{1}{\sqrt{K}} \sin^{-1} 1$$

$$t_1 = \frac{\pi}{2\sqrt{K}}$$

The period T of one complete oscillation is four times t_1.

$$T = \frac{2\pi}{\sqrt{K}}$$

If I is the moment of inertia of the pendulum with respect to the axis of rotation, the torque exerted on the pendulum by the supporting rod when the displacement is θ is

$$I\alpha = -IK\theta$$

as will be shown in Art. 130.

Let C_1 be the amount of the displacing couple necessary to produce a displacement of θ_1. Then

$$C_1 = IK\theta_1$$
$$K = \frac{C_1}{I\theta_1}$$

In terms of I, the expression for the period becomes

$$T = 2\pi\sqrt{\frac{\theta_1}{C_1}I}$$

Since for any given rod the ratio θ_1/C_1 is constant, it is seen that the period T varies as the square root of I. If another mass is added to the original mass of the pendulum to increase its moment of inertia to I_1, the period of vibration will be increased to T_1.

$$T_1 = T\sqrt{\frac{I_1}{I}}$$

EXAMPLE

A cast-iron plate 1 ft. in diameter and 2 in. thick is suspended by a steel rod 6 ft. long and 0.2 in. in diameter, for which E_s = 12,000,000 lb./sq. in., to form a torsion pendulum. What torque is required to displace the pendulum through an angle of 30°? If this torque is suddenly removed, what are the maximum angular acceleration and the maximum angular velocity? What is the period of oscillation?

Solution.

$$M = \frac{\pi \times 450}{4 \times 6 \times 32.2} = 1.83$$
$$I = \tfrac{1}{2} \times 1.83 \times \tfrac{1}{4} = 0.229$$

From Strength of Materials,

$$\text{Torque} = \frac{E_sJ\theta}{l}$$

$$\text{Torque } C_1 = \frac{12{,}000{,}000\pi^2}{72 \times 6 \times 2 \times 10{,}000 \times 12} = 1.142 \text{ lb.-ft.}$$

$$K = \frac{C_1}{I\theta_1} = \frac{1.142 \times 57.3}{0.229 \times 30} = 9.53$$

$$\alpha = -9.53 \times \frac{30}{57.3} = 5.0 \text{ rad./sec.}^2$$

$$\omega_0 = \sqrt{9.53} \times \frac{30}{57.3} = 1.62 \text{ rad./sec.}$$

$$\text{Period } T = \frac{2\pi}{\sqrt{9.53}} = 2.03 \text{ sec.}$$

Problems

1. If the supporting rod of the torsion pendulum described in the foregoing example is made twice as long, how is the period of vibration changed?
Ans. 1.414 times as great.

2. If the supporting rod of the torsion pendulum described in the foregoing example is made twice as large in diameter, how is the period of vibration changed? *Ans.* ¼ as great.

3. With two gear wheels of the same size attached symmetrically to the plate of the torsion pendulum described in the example above, the period of vibration was changed to 3.16 sec. Compute I of each gear wheel with respect to the axis of the torsion pendulum. *Ans.* $I = 0.163$.

129. Effective Forces on a Rotating Body.—Let the body shown in Fig. 367 represent any rotating body, F the resultant force causing rotation, and O the axis of rotation. Let P be any particle of the body of mass dM, at a radial distance of ρ from O. Then if the body has an angular velocity ω and an angular acceleration α, the tangential and normal components of the acceleration a are $a_t = \rho\alpha$ and $a_n = \rho\omega^2$, respectively, for each particle of mass dM. The effective force for each particle of mass dM is given by its two components, $dM\rho\alpha$ tangential to its path and $dM\rho\omega^2$ normal to its path. It will be seen that for particles of equal mass dM, the tangential effective forces vary directly with ρ and are always at right angles to it in the direction of the angular acceleration. The normal effective forces vary directly with ρ and are always directed along the radius toward the axis.

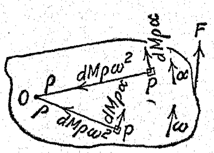


FIG. 367.

130. Moment of Tangential Effective Forces.—In Fig. 368, let O be the axis of rotation, C the center of gravity, F the result-

ant of all the external forces except the reaction of the axis O, and d the distance from the axis of rotation to the line of action of force F. The tangential effective force upon particle P of mass dM is $dM\rho\alpha$ normal to radius ρ as shown. (Since rotation alone is being considered, forces parallel to the axis of rotation are neglected.)

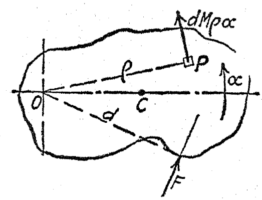


FIG. 368.

By the principle of Art. 109, the actual impressed forces could be replaced by the system of effective forces. Since neither the normal effective forces (not shown) nor the reaction at O (not shown) have any moment about the axis O, the moment of the tangential effective forces must be equal to the moment of the impressed forces, or

$$Fd = \int dM\rho^2\alpha$$

Since at any instant all particles of the body have the same value of α

$$Fd = \alpha \int dM\rho^2$$

The value of $\int dM\rho^2$ if integrated between the proper limits is I, the moment of inertia of the body with respect to the axis of rotation. Then

$$\mathbf{Fd = I\alpha}$$

The analogy of this equation to the equation $F = Ma$ was discussed in the footnote to Art. 82.

EXAMPLE 1

Figure 369 represents a cast-iron cylinder 3 ft. in diameter and 6 in. thick, free to rotate about its geometric axis O. If a force of 100 lb. is applied to a cord wrapped around the cylinder, what are the angular acceleration, the tangential acceleration, the angular velocity after 5 sec., and the number of revolutions that it has turned? Neglect the axle friction.

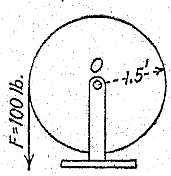


FIG. 369.

Solution.—The forces acting upon the cylinder are its weight, the reactions of the supports at O, and the force F. The first two forces have no moment about the geometric axis, so the moment

$$Fd = 100 \times 1.5 = 150 \text{ lb.-ft.}$$

$$I_0 \text{ of the cylinder} = \tfrac{1}{2}Mr^2 = 55.6$$

$$
\begin{aligned}
Fd &= I\alpha \\
150 &= 55.6\ \alpha \\
\alpha &= 2.698 \text{ rad./sec.}^2 \\
a_t &= r\alpha = 4.047 \text{ ft./sec.}^2 \\
\omega &= \alpha t, \quad \text{so } \omega_5 = 13.49 \text{ rad./sec.} \\
\theta &= \tfrac{1}{2}\alpha t^2 = 33.73 \text{ rad. in 5 sec.} \\
33.73 \div 2\pi &= 5.365 \text{ rev. in 5 sec.}
\end{aligned}
$$

EXAMPLE 2

Instead of a force $F = 100$ lb., let a weight of 100 lb. be hung from the cord in Example 1. Determine the angular acceleration and the tension T in the cord.

Solution.—First, let the cylinder be considered as the free body.

The equation $Fd = I\alpha$ gives $T \times 1.5 = 55.6\ \alpha$.

This equation contains two unknown quantities, so tne equation of motion of the suspended weight must be written.

$$100 - T = \frac{100}{32.2}a$$

Since the acceleration a of the weight is equal to the tangential acceleration a_t of the rim of the cylinder, and $a_t = r\alpha$,

$$
\begin{aligned}
100 - T &= \frac{100}{32.2}\ 1.5\alpha \\
\alpha &= 2.396 \text{ rad./sec.}^2 \\
T &= 88.82 \text{ lb.}
\end{aligned}
$$

Problems

1. Solve Example 1 if the cord by which the 100-lb. force is applied is wrapped around the axle 2 in. in diameter.

Ans. $\alpha = 0.15$ rad./sec.^{2}; $a_t = 0.225$ ft./sec.^{2}; $\omega_5 = 0.75$ rad./sec.; 0.299 rev.

2. Let the material of the cast-iron cylinder in Fig. 369 be the same in amount and outside diameter but composed of rim, spokes, and hub so arranged that $k = 1.3$ ft. Solve for α and T if a weight of 100 lb. is hung from a cord wrapped around the wheel, as in Example 2.

Ans. $\alpha = 1.66$ rad./sec.^{2}; $T = 92.3$ lb.

3. A cast-iron cylinder 4 ft. in diameter and 2 ft. long is rotating at 480 r.p.m. Upon a brake which rubs against the curved surface of the cylinder is a normal pressure of 600 lb. The coefficient of friction between the brake and the cylinder is $\frac{1}{3}$. If the friction at the bearings is neglected, what is the time required to bring the cylinder to rest? Through how many revolutions will it turn? *Ans.* $t = 88.3$ sec.; 353 rev.

4. A cast-iron flywheel 10 ft. in diameter has a rim 18 in. wide and 3 in. thick, and six spokes with elliptic cross sections with axes 5 in. and 3 in. To its axle is fastened a concentric drum 20 in. in diameter around which a cord is wrapped. If a 600-lb. weight is hung from the cord, what are the angular acceleration of the flywheel and the tension in the cord? Consider each pair of spokes as a slender rod passing through from rim to rim, and

neglect the moment of inertia of the hub, axle, and drum. Neglect also the axle friction. *Ans.* $\alpha = 0.122$ rad./sec.^{2}; $T = 598$ lb.

131. Resultant of Tangential Effective Forces.—In Fig. 370, let O be the axis of rotation, C the center of gravity, P any particle of mass dM, ρ the distance of particle P from the axis, and θ the angle between this radius and the line from the axis of rotation to the center of gravity. Let the line of OC be taken as the X axis. The tangential effective force acting on the particle P is $dM\rho\alpha$. Let this force be broken up into its X and Y components. Its X component is $dM\rho\alpha \sin\theta$, and its Y component is $dM\rho\alpha\cos\theta$.

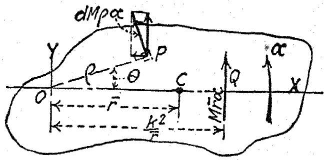


FIG. 370.

$$\Sigma F_X = \int dM\rho\alpha \sin\theta = \alpha\int y\,dM = \alpha M\bar{y}$$

Since $\bar{y} = 0$, $\Sigma F_X = 0$.

$$\Sigma F_Y = \int dM\rho\alpha\cos\theta = \alpha\int x\,dM = \alpha M\bar{x}$$

Since $\bar{x} = \bar{r}$, and $\Sigma F_X = 0$, the resultant of the tangential effective forces is $M\bar{r}\alpha$.

The resultant moment of the effective forces is equal to $I\alpha$ as shown in Art. 130. By the principle of moments, the moment of the resultant $M\bar{r}\alpha$ must equal the sum of the moments of its components. Let x by the distance from O to the line of action of the resultant $M\bar{r}\alpha$. Then

$$M\bar{r}\alpha x = I\alpha = Mk^2\alpha$$

$$x = \frac{k^2}{\bar{r}}$$

The resultant of the tangential effective forces on a body is a force equal to $M\bar{r}\alpha$, acting normal to a line joining the axis of rotation and the center of gravity, at a distance $k^2/\bar{r}$ from the axis of rotation.

It is evident from the foregoing discussion that, if the effective tangential force $M\bar{r}\alpha$ were reversed in direction and made to act tangentially through point Q, its moment would balance the accelerating moment of the external system of forces and would produce a static condition of rotation.

If a body rotates about its center of gravity so that points O and C coincide, $\bar{r} = 0$, so $k^2/\bar{r}$ = infinity. This shows that in this case the equivalent moment can be given only by a couple of moment $I\alpha$ and not by a single force.

EXAMPLE

A slender rod of length l is released from a horizontal position and allowed to rotate under the influence of gravity alone about a horizontal axis through one end, perpendicular to the axis of the rod. Determine the amount and position of the tangential effective force $M\bar{r}\alpha$ at the instant of starting.

Solution.—Let W be the weight of the rod. From the equation,

$$\text{Torque} = I\alpha$$

$$\frac{Wl}{2} = \frac{W}{g}\frac{l^2}{3}\alpha$$

$$\alpha = \frac{3g}{2l}$$

$$M\bar{r}\alpha = \frac{W}{g} \times \frac{l}{2} \times \frac{3g}{2l} = \frac{3}{4}W$$

$$\frac{k^2}{\bar{r}} = \frac{l^2}{3} \div \frac{l}{2} = \frac{2}{3}l$$

Problems

1. If the slender rod referred to in the example is a steel rod 1 in. in diameter and 6 ft. long, what is the value of the resultant tangential effective force after it has rotated through an angle of 30°? An angle of 90°?

Ans. 10.4 lb.; 0.

2. Solve Prob. 1 if the axis of rotation is 1 ft. from the end. *Ans.* 7.93 lb.; 0.

3. The rotating body shown in Fig. 371 consists of a semicircular steel plate with 6 in. radius and thickness of 2 in., connected to the axis of rotation O by a steel plate 8 in. long, 2 in. wide, and 1 in. thick. If released from rest in the position shown, compute the value of α, $M\bar{r}\alpha$, and $k^2/\bar{r}$.

Ans. α = 33.8 rad./sec.^{2}; $M\bar{r}\alpha$ = 31.4 lb.; $k^2/\bar{r}$ = 0.95 ft.

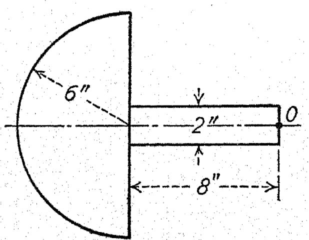


FIG. 371.

132. Resultant of Normal Effective Forces.—As stated in Art. 129, the normal effective forces for the particles of a rotating body at any instant are directly proportional to their radii and act toward the axis of rotation (see Fig. 367). In general, the resultant of these normal forces for the whole body is a force and a couple, as was shown in Art. 57. The solution of this problem in the general case is involved and difficult, for usually the resultant force does not act through the center of gravity, and the resultant

couple is hard to obtain, since it involves the product of inertia of the body. Fortunately, nearly all engineering problems in rotation come under a few special cases in which the value of the couple is zero or is easily obtained and the resultant force is easily located. Three of these cases will be discussed.

Case 1.—If a body has a plane of symmetry and rotates about any axis normal to this plane, the resultant normal effective force acts radially through the center of gravity and is equal to $M\bar{r}\omega^2$.

Let Fig. 372 (*a*) represent a body that has a plane of symmetry *QMNP*. Let the axis *OZ* normal to the plane of symmetry be the

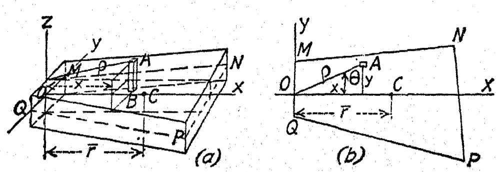


Fig. 372.

axis of rotation, and let the *X* axis be taken through the axis of rotation *O* and the center of gravity *C*. Let *AB* be any elementary prism of mass *dM* parallel to the axis *OZ*. Since each part of the prism has the same normal acceleration $a_n = \rho\omega^2$, the resultant of the normal effective forces for the prism *AB* is $dM\rho\omega^2$ acting in the plane of symmetry toward the axis *OZ*. Figure 372 (*b*) shows the section cut by the plane of symmetry. The *X* component of the force $dM\rho\omega^2$ is $dM\rho\omega^2 \cos\theta = dM\rho\omega^2\dfrac{x}{\rho} =$ $dM\omega^2 x$; and for the entire body $\Sigma F_x = \omega^2\int dMx = M\bar{x}\omega^2 = M\bar{r}\omega^2$. The *Y* component of the force $dM\rho\omega^2$ is $dM\rho\omega^2 \sin\theta = dM\rho\omega^2\dfrac{y}{\rho}$ $= dM\omega^2 y$; and for the entire body, $\Sigma F_y = \omega^2\int dMy = M\bar{y}\omega^2 = 0$, since $\bar{y} = 0$. Hence the resultant of all the normal effective forces is $M\bar{r}\omega^2$ acting in the plane of symmetry parallel to the *X* axis. Since the normal effective forces all pass through the axis *OZ*, they have no moment about *OZ*; hence their resultant can have no moment about *OZ* and must therefore lie in the axis *OX* through the center of gravity.

If the axis *OZ* passes through the center of gravity, $\bar{r} = 0$, so the resultant normal effective force $M\bar{r}\omega^2 = 0$.

Case 2.—If a body has a line of symmetry and rotates about an axis parallel to this line, the resultant normal effective force acts through the center of gravity and is equal to $M\bar{r}\omega^2$.

In Fig. 373, let AB be a line of symmetry of the body shown, and let the body be rotating about axis OZ parallel to AB. Consider the plate EF of thickness dz and mass dM, whose plane is normal to the axis OZ. By Case 1, the resultant normal effective force on plate EF is $dM\bar{r}\omega^2$, acting through the center of gravity of the plate toward the axis OZ. On each similar plate there is a corresponding normal effective force directed from line AB normal to axis OZ, and each force is proportional to the mass of its plate. As shown in the last paragraph of Art. 71, the resultant

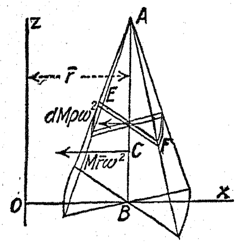


FIG. 373.

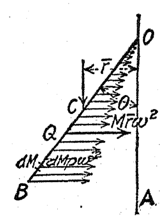


FIG. 374.

of this system of parallel forces acts through the center of gravity of the body. Its amount is equal to $\Sigma dM\bar{r}\omega^2 = M\bar{r}\omega^2$, since $\bar{r}$ and ω are constants.

If the axis OZ coincides with the line of symmetry, $\bar{r} = 0$ and the resultant normal effective force $M\bar{r}\omega^2 = 0$.

Case 3.—If a slender prismatic rod of length l is rotating about an axis through one end at any angle θ with the axis, the resultant normal effective force is equal to $M\bar{r}\omega^2$ and acts through a point distant $\frac{2}{3}l$ from the point of support.

In Fig. 374, let OB be the rod of length l with its center of gravity at C, and let it be rotating about the axis OA with angular velocity ω. Let the rod be divided into equal elementary parts. The normal effective force on any elementary mass dM is equal to $dM\rho\omega^2$, ρ being the radial distance of the mass dM. These forces are proportional to the radial distances of the masses and are acting normal to the axis AO. Since ω^2 is the same for all elements, the resultant of all the effective forces becomes $\omega^2\int dM\rho = M\bar{r}\omega^2$ and acts at point Q, distant $\frac{2}{3}l$ from O and

$k^2/\bar{r}$ from the axis OA. If the axis does not pass through the end of the rod, the part on each side of the axis is treated independently.

If the plane of symmetry of the body shown in Fig. 372 is not normal to the axis of rotation OZ, the resultant normal effective force no longer acts through the center of gravity but acts through the point in the plane of symmetry distant $k^2/\bar{r}$ from the axis OZ, and the couple mentioned in the first paragraph is induced. Likewise, if the line of symmetry of the body shown in Fig. 373 is not parallel to the axis of rotation OZ, the resultant normal effective force no longer acts through the center of gravity, but through the point distant $k^2/\bar{r}$ from the axis. This point may therefore be considered to be the fixed application point of the system. In special cases 1 and 2, the resultant effective force acts through the center of gravity as well as through this fixed point Q, and the moment of the couple is zero.

Problems

1. If the composite plate shown in Fig. 371 is rotating about an axis through O normal to the plate at a speed of 180 r.p.m., what is the amount of the normal effective force? *Ans.* 328 lb.

2. A cast-iron hemisphere 6 in. in diameter is rotating about a diameter of the base at a speed of 60 r.p.m. Compute the normal effective force. *Ans.* 1.69 lb.

3. Figure 375 represents the cast-iron hemisphere described in Prob. 2 connected to the vertical axis OZ by the horizontal rod OA, the mass of which may be neglected. Compute the speed at which the hemisphere must rotate about axis OZ in order that there shall be no bending moment in the rod OA at O. *Ans.* 177 r.p.m.

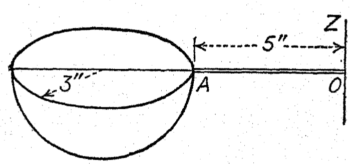


FIG. 375.

4. A slender rod 5 ft. long weighing 10 lb. is rotating about a vertical axis through the rod 1 ft. from the upper end at an angle of 30° with the rod. If the rod is rotating at a speed of 30 r.p.m., what is the amount of the induced couple at the support? What is the amount of the induced couple if it is rotating at a speed of 120 r.p.m.? At what speed must it rotate so that there is no induced couple?

Ans. 1.75 lb.-ft. outward; 84.5 lb.-ft. inward; 34.2 r.p.m.

133. Simple Circular Pendulum.—A *simple circular pendulum* consists of a particle vibrating in the arc of a vertical circle under the influence of gravity and some constraining radial force. The ideal simple circular pendulum may be closely approximated by means of a small heavy sphere at the end of a light cord.

Let A, Fig. 376, be such a body suspended by cord OA, of length l, and let distance along the arc be measured from C, positive to the right. The only force in the direction of motion is $-W \sin \theta$. From

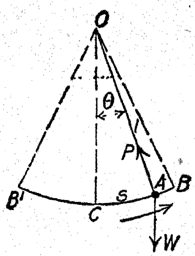


FIG. 376.

$F = \frac{W}{g}a,$

$$a_t = -g \sin \theta$$
$$v\,dv = a\,ds = -g \sin \theta\,ds$$

The expression for t in terms of the integral of $\sin \theta\,ds$ is a complicated elliptic form, but an approximate solution is comparatively simple and for vibrations of small amplitude is very slightly in error. For small values of θ, $\sin \theta = \theta$, approximately, so the foregoing equations become

$$a_t = -g\theta = -\frac{g}{l}s, \left(\text{since } \theta = \frac{s}{l}\right)$$

and

$$v\,dv = -\frac{g}{l}s\,ds$$

Let v_0 be the velocity at C, and v the velocity at A. Then

$$\int_{v_0}^{v} v\,dv = -\frac{g}{l}\int_0^s s\,ds$$
$$v^2 - v_0^2 = -\frac{g}{l}s^2$$

Since $v = 0$ when $s = s_B$,

$$v_0^2 = \frac{g}{l}s_B^2$$

The insertion of this value of v_0^2 in the equation above gives

$$v^2 = \frac{g}{l}(s_B^2 - s^2)$$
$$v = \sqrt{\frac{g}{l}}\sqrt{s_B^2 - s^2}$$

Since

$$v = \frac{ds}{dt}$$
$$dt = \sqrt{\frac{l}{g}}\frac{ds}{\sqrt{s_B^2 - s^2}}$$

If time is measured from the instant the pendulum is at C, moving to the right, this becomes

$$\int_0^t dt = \sqrt{\frac{l}{g}}\int_0^s \frac{ds}{\sqrt{s_B^2 - s^2}}$$

$$t = \sqrt{\frac{l}{g}}\sin^{-1}\frac{s}{s_B}$$

To get the time required for the pendulum to move from C to B, let $s = s_B$. Then

$$t_B = \frac{\pi}{2}\sqrt{\frac{l}{g}}$$

To get the time required for the pendulum to move from C to B and back to C, let $s = 0$. Then

$$t_C = \pi\sqrt{\frac{l}{g}}$$

The time required for the pendulum to move from C to B is therefore the same as that to move from B to C.

Motion to the left of C exactly corresponds to motion to the right of C, so the time of one complete period of vibration

$$\mathbf{T} = 2\pi\sqrt{\frac{\mathbf{l}}{\mathbf{g}}}$$

This equation for T is independent of θ, so the time of vibration is independent of the amplitude for small values of θ.

It will be seen from the equation $a_t = -\frac{g}{l}s$ that for vibrations of small amplitude the acceleration is proportional to the displacement, and so the motion is practically simple harmonic motion.

Since the ball of the pendulum is accelerated toward the center with an acceleration v^2/l, the summation of forces normal to the path gives

$$\Sigma F_n = P - W\cos\theta = \frac{W}{g}\frac{v^2}{l},$$

P being the tension in the cord.

$$P = W\cos\theta + \frac{W}{g}\frac{v^2}{l}$$

Problems

1. Compute the length of the simple pendulum that will make a complete vibration in $\frac{1}{5}$ sec.; in 1 sec.; in 5 sec. *Ans.* 0.391 in.; 9.78 in.; 20.4 ft.

2. A mine cage is suspended by a cable 600 ft. long. Compute the time for one complete vibration. *Ans.* 27.1 sec.

3. A girder weighing 18,000 lb. is suspended by a cable 120 ft. long. What horizontal pull is necessary to hold it 6 ft. from the vertical position? What is the tension in the cable as the girder is allowed to swing back through its vertical position? *Ans.* 901 lb.; 18,045 lb.

134. Compound Pendulum.—Any physical body suspended from a horizontal axis not passing through the center of gravity and free to rotate under the influence of gravity and the reaction of the support is called a *compound pendulum.*

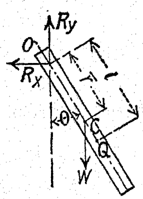


FIG. 377.

Let Fig. 377 represent a compound pendulum of weight W, suspended at O, and let C be its center of gravity. Let I be its moment of inertia with respect to the axis of rotation, k its radius of gyration, $\bar{r}$ the distance from the support to the center of gravity, and α the angular acceleration. The equation of moments about O gives

$$\Sigma M_o = -W\bar{r} \sin \theta = I\alpha$$

Since

$$I = Mk^2 = \frac{W}{g}k^2$$

$$\alpha = -\frac{W\bar{r} \sin \theta}{\frac{W}{g}k^2} = -\frac{\bar{r}g \sin \theta}{k^2}$$

The tangential acceleration a_t of point Q, at a distance l from the axis O, is

$$a_t = l\alpha = -\frac{\bar{r}lg \sin \theta}{k^2}$$

If the length l is taken equal to $k^2/\bar{r}$, the tangential acceleration a_t of point Q will be

$$a_t = -g \sin \theta$$

which is the same as the acceleration of the simple circular pendulum. It is seen from this that a simple circular pendulum of length $l = k^2/\bar{r}$ will vibrate in the same time as the compound pendulum. The length $k^2/\bar{r}$ is called the *length of the compound pendulum,* and the point Q is called the *center of oscillation.*

Since $l = k^2/\bar{r}$, the time of one complete period becomes

$$T = 2\pi\sqrt{\frac{k^2}{g\bar{r}}}$$

Also,

$$k = \frac{T}{2\pi}\sqrt{g\bar{r}}$$

The point of suspension and the center of oscillation are interchangeable, as will now be shown. Let k_c be the radius of gryation of the pendulum with respect to the axis through the center of gravity C parallel to the axis of rotation. Since

$$k_c{}^2 = k^2 - \bar{r}^2 \quad \text{and } k^2 = \bar{r}l$$
$$k_c{}^2 = \bar{r}(l - \bar{r}) = OC \times QC$$

Since k_c is a constant, regardless of the point of suspension, the product $OC \times QC$ must be a constant. If OC is made smaller, QC becomes proportionately larger, and vice versa.

Again, this equation would not be altered in any way if the positions of O and Q were interchanged; hence if the pendulum is inverted and suspended from Q, point O must become the center of oscillation. Since the length l remains the same, the time of vibration is the same.

EXAMPLE

A cylinder 2 in. in diameter and 12 in. long is hung from an axis through the diameter of one end. Find the time of oscillation. From what other point could it be suspended to vibrate in the same length of time?

Solution.

$$k^2 = \frac{r^2}{4} + \frac{h^2}{3} = 0.335 \text{ ft.}^2$$

$$T = 2\pi\sqrt{\frac{0.335}{32.2 \times 0.5}} = 0.905 \text{ sec.}$$

The center of oscillation is given by

$$\frac{k^2}{\bar{r}} = \frac{0.335}{0.5} = 0.67 \text{ ft.}$$

If the cylinder is suspended from the center of oscillation, it will vibrate in the same length of time.

Problems

1. Find the time of oscillation and the center of oscillation of the cylinder described in the example above if it is suspended from an axis through the cylinder 5 in. from the end and normal to the geometric axis.

Ans. 1.16 sec.; 1.104 ft.

2. A plate 4 in. wide and 1 in. thick (Fig. 378) is to vibrate as a pendulum about axis OZ, 1 in. from the upper end. Get the length h so that it will make a complete vibration in 1 sec. Locate the other points from which it may be suspended and have the same period of vibration.

Ans. 1.276 ft.; 1 in. from lower end; 4.59 in. from either end.

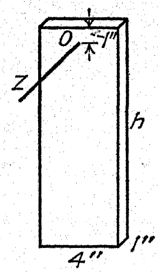


FIG. 378.

135. Conical Pendulum.—The *conical pendulum* represented in Fig. 379(a) consists of a small body A suspended by a cord from point O on a vertical axis about which it rotates in a horizontal plane. Point C is the center of rotation of body A. If the angle θ with the axis is constant, the speed is constant and $a_t = dv/dt = 0$. The tangential force $F_t = Ma_t = 0$. If the rotating body is subjected to air resistance, it is necessary that a small positive force just equal to the negative air resistance shall be acting in order to maintain a constant speed.

In the normal plane, the vertical plane through the cord, there are only two forces acting upon the body, the weight W and the tension P in the cord, as shown in Fig. 379(b). Under the action of these two forces, the body has an acceleration $a_n = v^2/r$ toward the center C about which it is rotating. If the reversed effective force $\frac{W}{g}\frac{v^2}{r}$ is added to the free body, as in Fig. 379(c), the body is under static conditions and all the equations of equilibrium are true. The equation of moments with respect to point O gives

$$\frac{W}{g}\frac{v^2}{r}h - Wr = 0$$

$$v = r\sqrt{\frac{g}{h}}$$

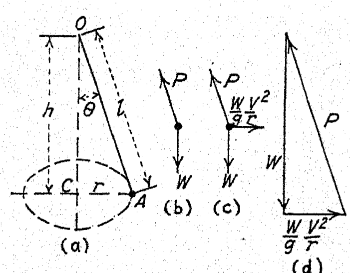


FIG. 379.

This equation gives the speed v necessary to keep the pendulum at the constant angle θ with the axis. The force triangle is shown in Fig. 379(d), from which the value of P is obtained.

$$P = \sqrt{W^2 + \left(\frac{W}{g}\frac{v^2}{r}\right)^2}$$

The period T of rotation is given by the expression

$$T = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{h}{g}}$$

The number of revolutions per second, or the frequency, is

$$n = \frac{1}{T}$$

$$n = \frac{1}{2\pi}\sqrt{\frac{g}{h}}$$

As n becomes smaller, h becomes larger, until when $\theta = 0$, $h = l$. For this value,

$$n = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

As n increases from zero, the expression just derived gives the limiting value for which lifting impends.

Since $h = gr^2/v^2 = g/4\pi^2n^2$, it is seen that h is independent of l and depends only upon the angular speed. If conical pendulums of different lengths are rotated at the same number of revolutions per second, they will all have the same height h.

Problems

1. Compute the value of n when lifting impends for a conical pendulum 1 ft. long; for one 10 ft. long. *Ans.* 0.904; 0.286.

2. A weight of 3 lb. is suspended with its center of gravity 5 ft. from the point of support. Compute the tangential velocity v necessary to hold it at an angle of 60° with the vertical axis of rotation. Compute the frequency n and the tension P in the cord.

Ans. $v = 15.52$ ft./sec.; $n = 0.572$ rev./sec.; $P = 6$ lb.

3. Compute the angle θ, the velocity v, and the tension P in the cord if the conical pendulum described in Prob. 2 is rotating at 1 rev./sec.

Ans. $\theta = 80°35'$; $v = 31$ ft./sec.; $P = 18.4$ lb.

136. Weighted Conical Pendulum Governor.—Figure 380(a) represents a weighted conical pendulum governor which consists of two spheres A, A, at the ends of arms BA, and a weight W_1 supported by the collar CC. Consider first the weight W_1 as the free body, Fig. 380(b), with the governor rotating uniformly. The body is under static conditions, so equation $\Sigma F_y = 0$ gives

$$P = \frac{W_1}{2 \cos \theta_1}$$

Consider next one of the spheres and its arm as the free body, Fig. 380(*c*). The impressed forces acting upon the free body are three in number, the weight W, the tension P, and the pin reaction at B. If the effective force $M\bar{r}\omega^2$ is added to the system reversed in direction, the free body will be under static conditions. Accurately, the force $M\bar{r}\omega^2$ should act through the center of oscillation of the sphere and its arm, and the weight of the arm should be considered, but the error is small if the weight of the rod is neglected and $M\bar{r}\omega^2$ is considered to act through the center of gravity of the sphere. The equation $\Sigma M_B = 0$ gives

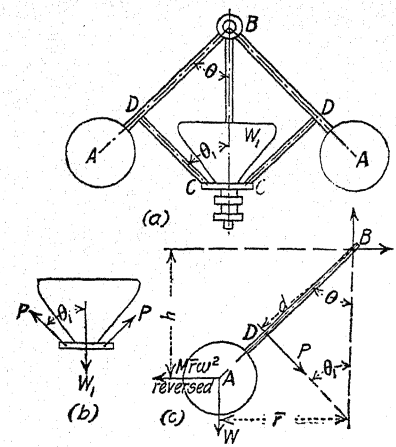


FIG. 380.

$$Pd + W\bar{r} = M\bar{r}\omega^2 h$$

Problems

1. In Fig. 380, let $BD = 12$ in., $DA = 4$ in., and $\theta = \theta_1 = 45°$ in the lowest position. Let spheres A, A be cast iron, 5 in. in diameter, and let the weight W_1 be a cast-iron cone, with radius of base 4.5 in. and altitude 7.5 in. Neglecting the weight of the arms, compute the speed at which the governor begins to act. *Ans.* 93.7 r.p.m.

2. If the governor described in Prob. 1 has a spring that carries 20 lb. of the weight of W_1 when it is in its lowest position, at what speed will it begin to act? *Ans.* 78 r.p.m.

3. If the distance CC, Fig. 380, is 3 in., what speed of rotation will be necessary to make the angle $\theta = 60°$? *Ans.* 115 r.p.m.

137. Superelevation of Railway Track.—If both rails of a railroad track are on the same level on a curve and a car moves around the curve, there is a heavier vertical pressure on the car from the rail on the outside of the curve. As the speed is increased, a value will be reached for which the vertical pressure of the outer rail is equal to the weight of the car, and that of the inner rail is zero. This is the limiting value of the speed possible without overturning.

In order to increase the range of speed possible on a curve, the track is built with the outer rail higher than the inner one. It

is customary to select a desired mean speed for a given curve and to give the outer rail the superelevation necessary for equal pressures from the inner and outer rails. For any speed above this, up to the limiting overturning speed, the pressure on the car from the outer rail is greater than that from the inner rail. There is also a resultant flange pressure parallel to the ties acting inward on the car. For any speed of the car less than this mean speed, the pressure from the inner rail is greater than that from the outer rail, and the resultant flange pressure parallel to the ties acts outward.

Figure 381 illustrates the conditions necessary for equal rail pressures. R_1 and R_2 are the equal pressures of the rails on the wheels, so their resultant R acts through the center of gravity C. W is the weight of the car, r is the radius of the curve and v is the speed of the car. Since r is horizontal, and the car is accelerated toward the center, the reversed effective force $\frac{W}{g}\frac{v^2}{r}$ must act norizontally outward through the center of gravity C. The conditions are now static conditions, and the three force vectors form a closed triangle, Fig. 381(b). If θ is the angle that the resultant R makes with the vertical,

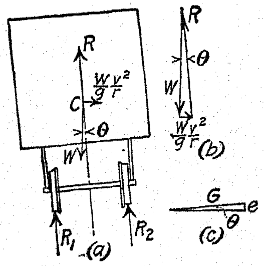


FIG. 381.

$$\tan\theta = \frac{v^2}{gr}$$

If G is the gage of the track, Fig. 381(c), and e the superelevation,

$$e = G\sin\theta$$

For small angles the sine and the tangent are approximately equal, so

$$e = \frac{Gv^2}{gr} \text{ (approx.)}$$

If a car is in a train, the drawbar pulls at the ends of the car are tangent to the curve at those points and therefore have small components parallel to the radius at the center of the car. If the car is moving faster than the mean speed for which the curve was designed, these components add to the stability of the car. On

the other hand, if the car is moving slower than the mean speed, these components act with the weight in tending to overturn the car inward and in increasing the resultant flange pressure outward parallel to the ties.

EXAMPLE

Determine the superelevation of the outer rail of a track of gage $G = 4.9$ ft. (center to center of rail) on a curve of radius $r = 2865$ ft. to give zero resultant flange pressure at a speed of 30 m.p.h. If a 100,000-lb. car with its center of gravity 5 ft. above the track has a speed of 60 m.p.h. on the curve, what is the pressure of the outer rail on the car normal to the ties?

Solution.—The speed is 44 ft./sec. The approximate expression may be used.

$$e = \frac{4.9 \times 44 \times 44}{32.2 \times 2865}$$
$$e = 0.103 \text{ ft.} = 1.236 \text{ in.}$$

For a speed of 88 ft./sec.,

$$\frac{W}{g}\frac{v^2}{r} = \frac{100{,}000 \times 88 \times 88}{32.2 \times 2865} = 8400 \text{ lb.}$$

The equation of moments with respect to R_1, Fig. 381, gives

$$4.9R_2 = 100{,}000 \times 2.3432 + 8400 \times 5.05$$
$$R_2 = 56{,}480 \text{ lb.}$$

Problems

1. If the car described in the example is moving around the curve at a speed of 10 m.p.h., what is the pressure of the outer rail on the car in a direction normal to the ties? What is the resultant flange pressure of the inner rail parallel to the ties? *Ans.* 48,060 lb.; 1870 lb.

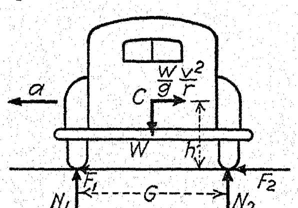


FIG. 382.

2. Compute the superelevation of the outer rail of a railway track on a 10° curve to give equal rail pressures on a car when it is moving around the curve at 20 m.p.h. If a car weighing 120,000 lb. with its center of gravity 4 ft. above the track has a speed of 45 m.p.h. on this curve, what is the pressure of the outer rail on the car normal to the ties? What is the resultant flange pressure parallel to the ties? (A 10° curve is one on which a chord of 100 ft. subtends an angle of 10° at the center.)

Ans. $e = 0.228$ ft.; 79,160 lb.; 22,750 lb.

138. Banking of Highway Curves.—If a car is moving with velocity v around a curve of radius r on a level roadway, the forces acting are as shown in Fig. 382. This is a rear view of the car, the center of the curve being to the left. The forces acting on the

car consist of its weight W, the normal reactions N_1 and N_2, and the frictional forces F_1 and F_2. In addition to the actual forces, the reversed effective force Wv^2/gr is also shown, acting through the center of gravity, C. Let $N_1 + N_2 = N$, and let $F_1 + F_2 = F$. Then, since $N = W$, slipping will impend when $F = fW = Wv^2/gr$, f being the coefficient of friction. The maximum velocity possible without skidding will be

$$v = \sqrt{fgr}$$

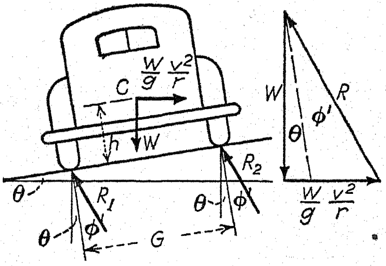
If the friction is great enough to prevent skidding, the car may overturn. As the velocity v increases, reaction N_1 becomes less and less; and when it reaches zero, overturning impends. At this speed, the moment equation with respect to N_2 gives

$$\frac{WG}{2} = \frac{Wv^2h}{gr}$$

$$v = \sqrt{\frac{Ggr}{2h}}$$

On highways intended for high-speed travel, curves are banked so that at some mean speed the reactions on the wheels are normal to the surface. This angle θ is given by the expression

$$\tan\theta = \frac{v^2}{gr}$$



(a) (b)

FIG. 383.

At a speed less than this mean speed, there will be a frictional reaction outward on the wheels, whereas at a speed greater than the mean speed there will be a frictional reaction inward on the wheels. As the speed is increased, skidding will impend when the resultant reactions R_1 and R_2, Fig. 383, act at the angle of static friction ϕ' with the normal and, therefore, at the angle $(\theta + \phi')$ with the vertical. If R is the resultant of R_1 and R_2,

$$\tan(\theta + \phi') = \frac{v^2}{gr}$$

as shown in the force triangle, Fig. 383(b).

As before, if the friction is large enough to prevent skidding, overturning will impend when the speed is great enough to reduce

R_1 to zero. The moment equation with respect to reaction R_2 will give the expression for this limiting speed. It is convenient to resolve the two forces at C into their components normal and parallel to the surface of the roadway.

$$\frac{Wv^2}{gr} \cos \theta \times h = \frac{Wv^2}{gr} \sin \theta \times \frac{G}{2} + W \cos \theta \times \frac{G}{2} + W \sin \theta \times h$$

EXAMPLE

A concrete highway curve with a radius of 500 ft. is superelevated so that there will be no lateral pressure for a vehicle speed of 30 m.p.h. If the coefficient of friction is $f = 0.2$, for what speed will skidding impend?

Solution.

$$\theta = \tan^{-1} \frac{44 \times 44}{32.2 \times 500}$$
$$\theta = 6°51'$$

The angle of friction with the normal is

$$\phi' = \tan^{-1} 0.2$$
$$\phi' = 11°21'$$

When skidding impends, $\tan(\theta + \phi') = v^2/gr$

$$\frac{v^2}{32.2 \times 500} = 0.32878$$
$$v^2 = 5293$$
$$v = 72.76 \text{ ft./sec.}$$
$$= 49.6 \text{ m.p.h.}$$

Problems

1. A highway curve with a radius of 300 ft. is banked so that there is no lateral pressure on the wheels at a speed of 25 m.p.h. What is the minimum coefficient of friction so that a car will not skid at 5 m.p.h.? What is the minimum coefficient of friction so that it will not skid at 50 m.p.h.? *Ans.* $\theta = 7°55'$; 0.133; 0.388.

2. With $f = 0.1$, what is the radius of the shortest curve in which a car may be turned on a level roadway without skidding if the speed is 15 m.p.h.? *Ans.* 150 ft.

3. With tread distance $G = 5$ ft. and height $h = 2$ ft., what is the radius of the shortest curve in which a car may be turned on a level roadway at a speed of 60 m.p.h. without overturning if friction is great enough to prevent skidding? If skidding is prevented by friction alone, what minimum coefficient of friction is required? *Ans.* $r = 192.4$ ft.; $f = 1.25$.

4. A highway curve with a radius of 400 ft. is banked so that there is no lateral pressure on the wheels of a car at a speed of 30 m.p.h. If a car weighing 4000 lb., with tread $G = 5$ ft. and height $h = 28$ in., travels around the curve at 60 m.p.h., what is the normal pressure on the outer

wheels? What is the total frictional reaction? If friction is great enough to prevent skidding, at what speed would overturning impend?

Ans. $\theta = 8°32'$; 2990 lb.; 1790 lb.; 93.2 m.p.h.

139. Centrifugal Tension in Flywheels.—If the tension in the arms of a flywheel is neglected, the tensile stress in the rim due to rotation may be computed. For the half rim shown in Fig. 384, the normal effective force $M\bar{r}\omega^2$ acts through the center of gravity. Also, the effective force reversed as indicated would be in equilibrium with the two induced tensile forces P, P. Then, if W is the weight of the half rim,

$$P = \frac{1}{2}\frac{W}{g}\bar{r}\omega^2$$

If r is the mean radius of the rim, $\bar{r} = 2r/\pi$ (approx.).

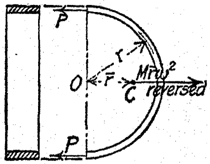


FIG. 384.

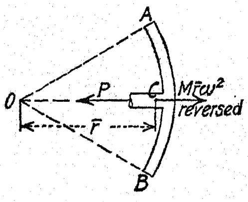


FIG. 385.

Let w be the weight per unit volume, or $w = W/\pi rA$. The unit stress

$$s = \frac{P}{A} = \frac{W\bar{r}\omega^2}{2gA} = \frac{Wr\omega^2}{gA\pi}$$

If ω^2 is replaced by v^2/r^2, and $W/\pi rA$ by w, this becomes

$$s = \frac{wv^2}{g}$$

Since w is in units of pounds per cubic foot, v in units of feet per second, and g in units of feet per second per second, s will be in units of pounds per square foot.

Large flywheels are sometimes cast in two parts and are bolted together at the hub and at the two places on the rim. In this case, the tensile stresses in the bolts are in equilibrium with the reversed effective force $M\bar{r}\omega^2$. In this expression, M is the mass of the complete half of the flywheel, and $\bar{r}$ is the distance to its center of gravity.

If the tension in the rim of the flywheel is neglected, the tensile stress in the arms may be computed. In Fig. 385, let AB be the

part of the rim carried by one arm. Let W be the weight of this part, and let $\bar{r}$ be the distance from O to its center of gravity. The induced tensile force caused by rotation is

$$P = \frac{W}{g}\bar{r}\omega^2$$

The unit stress is $s = P/A$, A being the area of the cross section of the arm.

In the usual case, both the rim and the arms take the centrifugal tensile forces, so that the stresses are something less than those computed above. The distribution of the stresses between the rim and the arms is indeterminate.

Problems

1. Show that, if the tension in the arms is neglected, the speed necessary to produce the same stress in the rims of flywheels varies inversely as their radii.

2. A cast-iron flywheel 12 ft. in diameter has a rim 2 in. thick and 12 in. wide. If the wheel is rotating at 300 r.p.m., what is the unit centrifugal tensile force in the rim if the tension in the arms is neglected?

Ans. 3350 lb./in.^{2}.

3. A flywheel weighing 3000 lb. is cast in two parts and is fastened together by 12 bolts. The value of $\bar{r}$ for each half of the wheel is 3 ft. If the bolts are ¾ in. in diameter, with a diameter of 0.62 in. at the root of the thread, and the allowable stress in the bolts is 15,000 lb./in.^{2}, what is the maximum allowable speed of rotation? *Ans.* 188 r.p.m.

4. If the flywheel described in Prob. 2 has six arms, the cross section of each being elliptical, with axes 4 and 2 in., respectively, what unit stress would be caused in the arms if each one carried its part of the rim?

Ans. 12,800 lb./in.^{2}.

140. Center of Percussion.—Let P, Fig. 386, be an impulsive force or blow that causes angular acceleration of the body suspended from O. The force P is variable; but at any instant during the blow,

$$Pd = I\alpha$$

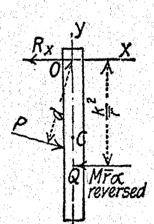


FIG. 386.

By the principle of Art. 131, the resultant tangential effective force $M\bar{r}\alpha$ acts at point Q, distant $k^2/\bar{r}$ from the point of support. Let R_x be the tangential component of the reaction at O caused by P. If the effective force $M\bar{r}\alpha$ is applied reversed, the system shown will be in equilibrium. By using the equation $\Sigma M_Q = 0$, the value of R_x may be computed for any value and position of

force P. It is evident that if force P is applied at point Q, $R_x = 0$. The point Q at which the body may be struck without producing any reaction parallel to the tangent is called the center of percussion. It is coincident with the center of oscillation of a pendulum. It follows, then, that the center of percussion and the point of suspension are interchangeable.

Problems

1. A slender rod 4 ft. long is suspended in a vertical position from an axis through the upper end. Get the amount and direction of the reaction at the support caused by a force of 100 lb. applied normal to the rod 1 ft. below the support; 2 ft. below the support; 3 ft. below the support.

Ans. 62.5 lb. opposite; 25 lb. opposite; 12.5 lb. in same direction.

2. Solve Prob. 1 if the point of support is 1 ft. below the upper end.

Ans. 57.1 lb. opposite; 14.3 lb. opposite; 28.6 lb. in same direction.

141. Reactions of Supports of Rotating Bodies.—In addition to the static reactions and the reactions caused by any external impressed forces, a rotating body may have also induced kinetic reactions. If, in addition to the weight and any other impressed forces, the normal and tangential effective forces are added to the system reversed in direction, the body is under static conditions, and any static equations of equilibrium may be written. These reversed effective forces act through a point at a distance $k^2/\bar{r}$ from the axis of rotation.

In making solution, the unknown reactions are commonly replaced by their rectangular components.

If $\bar{r} = 0$, $M\bar{r}\omega^2 = 0$ and $M\bar{r}\alpha = 0$, so there are no *kinetic* reactions of the supports for Cases 1 and 2, Art. 132, but the reactions are the same as when the body is at rest. In Case 3, if the axis passes through the center of gravity of the rod, the reaction of the support becomes a couple.

EXAMPLE 1

Figure 387(a) represents a steel disk 1 ft. in diameter and 1 in. thick, free to rotate about an element through O. If it starts from rest with C vertically above O and rotates under the influence of gravity alone, find the normal and tangential components of the hinge reaction at O when $\theta = 90°$.

Solution.—$\bar{r} = \frac{1}{2}$ ft.; $W = 32.07$ lb.; $M = 0.995$; $I_c = \frac{1}{2}Mr^2 = 0.124$; $I_0 = I_c + Mr^2 = 0.373$.

The equation of motion is

$$W\bar{r} \sin \theta = I_O\alpha$$

When $\theta = 90°$, $\sin\theta = 1$, and this equation becomes

$$32.07 \times 0.5 = 0.373\,\alpha$$
$$\alpha = 43 \text{ rad./sec.}^2$$
$$M\bar{r}\alpha = 0.995 \times 0.5 \times 43 = 21.4 \text{ lb.}$$

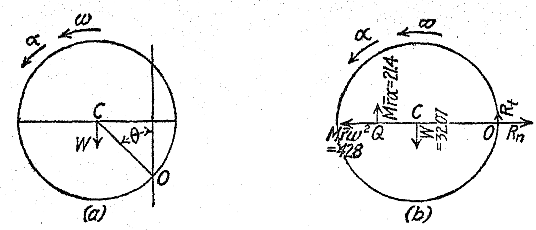


FIG. 387.

If the value of α from the foregoing equation of motion is substituted in the general equation $\omega\,d\omega = \alpha\,d\theta$, it becomes,

$$I\omega\,d\omega = W\bar{r}\sin\theta\,d\theta$$

$$I\int_0^{\omega}\omega\,d\omega = W\bar{r}\int^{\frac{\pi}{2}}\sin\theta\,d\theta$$

$$\tfrac{1}{2}I\omega^2 = -W\bar{r}\Big[\cos\theta\Big]_0^{\frac{\pi}{2}}$$

$$\omega^2 = \frac{2W\bar{r}}{I}$$

$$\omega^2 = 2 \times 32.07 \times \frac{0.5}{0.373} = 86$$

$$M\bar{r}\omega^2 = 0.995 \times 0.5 \times 86 = 42.8 \text{ lb.}$$

$$\frac{k^2}{\bar{r}} = \frac{I}{M\bar{r}} = 0.75 \text{ ft.} = \text{distance } OQ$$

Figure 387(*b*) shows the free-body diagram in the position asked for, with the effective forces reversed, and the hinge reaction represented by its normal and tangential components. From the static equations of equilibrium,

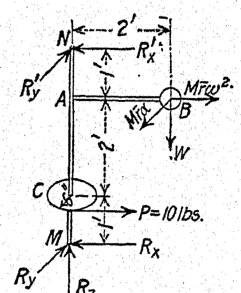


FIG. 388.

$$\Sigma F_x = 0, \quad \text{so} \quad R_n = 42.8 \text{ lb.}$$
$$\Sigma F_y = 0, \quad \text{so} \quad R_t = 32.07 - 21.4 = 10.67 \text{ lb.}$$

EXAMPLE 2

A vertical axle MN 4 ft. long carries a horizontal arm AB 2 ft. long which is attached to the axle 1 ft. from the top, as shown in Fig. 388. On the end of the horizontal arm is a cast-iron sphere 6 in. in diameter. The axle is rotated positively by a force of 10 lb. on the cord that passes around the pulley C. If the sphere starts from rest in the XZ plane and the force acts parallel to the X axis, solve for the reactions due to the sphere and the 10-lb. force after one revolution.

Solution.

$$W = \tfrac{4}{3}\pi r^3 \times 450 = 29.45 \text{ lb.}$$

$$M = 0.914; \ I_{MN} = 3.679; \ \frac{k^2}{\bar{r}} = 2.013 \text{ ft.}; \ \theta = 2\pi \text{ rad.} = 360°$$

Equation $Pd = I\alpha$ gives

$$10 \times 0.5 = 3.679\,\alpha$$
$$\alpha = 1.36 \text{ rad./sec.}^2$$

Since the acceleration is constant,

$$\omega = \sqrt{2\alpha\theta} = 4.135 \text{ rad./sec. after 1 rev.}$$
$$M\bar{r}\alpha = 0.914 \times 2 \times 1.36 = 2.486 \text{ lb.}$$
$$M\bar{r}\omega^2 = 0.914 \times 2 \times 17.1 = 31.25 \text{ lb.}$$

The last two forces are added reversed in Fig. 388, so the system as shown is in equilibrium.

Equation $\Sigma F_z = 0$ gives

$$R_z = W = 29.45 \text{ lb.}$$

Equation $\Sigma M_{Ry} = 0$ gives

$$(R_x' \times 4) - (29.45 \times 2) - (31.25 \times 3) - (10 \times 1) = 0$$
$$R_x' = 40.66 \text{ lb.}$$

Equation $\Sigma F_x = 0$ gives

$$R_x = 0.59 \text{ lb.}$$

Equation $\Sigma M_{Rx} = 0$ gives

$$(R_y' \times 4) - (2.486 \times 3) = 0$$
$$R_y' = 1.86 \text{ lb.}$$

Equation $\Sigma F_y = 0$ gives

$$R_y = 0.62 \text{ lb.}$$

Problems

1. With the same general data as in Example 1, compute the vertical and horizontal components of the reaction when $\theta = 45°$.

Ans. $R_y = 12.51$ lb. upward; $R_x = 1.83$ lb. to the left.

2. With the same general data as in Example 1, solve for the angle θ at which the horizontal component of the reaction changes direction.

Ans. 48°10′.

3. With the same general data as in Example 2, solve for the reactions when the sphere has rotated $\tfrac{3}{4}$ revolution from rest.

Ans. $R_x = 6.88$ lb.; $R_x' = 0.64$ lb.; $R_y = -8.88$ lb.; $R_y' = 32.28$ lb.

4. With the same general data as in Example 2, solve for the reactions when the sphere has rotated 2 sec. from rest.

Ans. $\theta = 2.72$ rad.; $R_x = 18.08$ lb.; $R_x' = -19.34$ lb.; $R_y = 4.09$ lb.; $R_y' = -11.88$ lb.

5. A cast-iron cone has an altitude of 12 in. and radius of base of 4 in. It is free to rotate about a horizontal axis normal to the geometric axis at its middle point. If released from rest with the center of gravity vertically above the axis of rotation, what are the normal and tangential components of the reaction after it has rotated 120°?

Ans. $R_n = 110.18$ lb.; $R_t = 21.05$ lb.

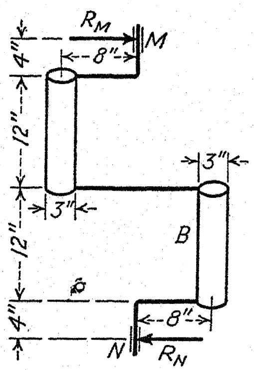


FIG. 389.

6. In Fig. 389, A and B are steel cylinders 3 in. in diameter and 1 ft. long rotating about the vertical axis MN. Compute the kinetic reactions R_M and R_N when they are rotating at 600 r.p.m.

Ans. $R_M = R_N = 735$ lb.

142. Balancing of Rotating Bodies.—It was seen in Art. 141 that if a body rotates about an axis not through its center of gravity, the bearing reactions have kinetic components. These continually change in direction and so cause destructive vibration.

Also, even though the center of gravity may be in the axis of rotation, if the two parts of the body into which it may be divided by the plane through the center of gravity normal to the axis are not symmetrical, a kinetic couple is induced at the reactions. This was illustrated in Prob. 6, Art. 141.

Balancing consists in adding rotating parts in such a way that the effective forces for the entire system are in equilibrium and

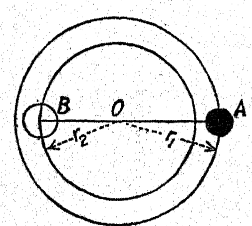


FIG. 390.

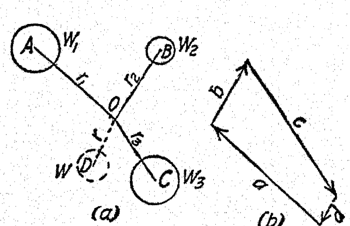


FIG. 391.

no kinetic reactions are induced. The static reactions due to gravity and any other constant impressed forces remain constant whether the body is at rest or in motion.

In the following discussion, only rotation at constant speed will be considered. Let A, Fig. 390, be a body of weight W_1, at a radial distance r_1, rotating about the axis through O normal to OA with

angular velocity ω. By its rotation it exerts upon the axis O a centrifugal pull equal to $W_1r_1\omega^2/g$ which continually changes in direction and causes variable reactions at the supports. If, however, another body of weight W_2 is placed diametrically opposite in the plane of rotation at the end of radius r_2, of such length that

$$\frac{W_2r_2\omega^2}{g} = \frac{W_1r_1\omega^2}{g}$$

the two bodies are in balance, since the centrifugal pulls are equal and opposite. Since ω and g are constants, the condition above reduces to

$$W_2r_2 = W_1r_1$$

It should be noted that the condition $W_2r_2 = W_1r_1$ is also the condition for static balance. In practice it is customary to deter-

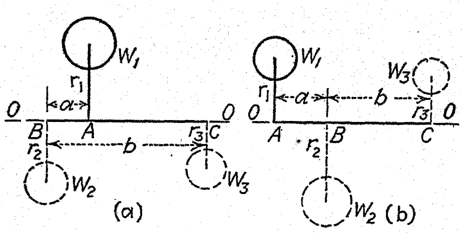


FIG. 392.

mine the necessary value of W_2r_2 by means of static balancing.

Figure 391(a) represents a number of bodies W_1, W_2, W_3 at radial distances r_1, r_2, r_3 in the same plane, rotating about the axis through O normal to their plane. These are to be balanced by a single weight W with radius r. If a vector polygon, Fig. 391(b), is drawn in which a, b, and c represent W_1r_1, W_2r_2, and W_3r_3, respectively, in magnitude and direction, the closing line d will represent Wr in magnitude and direction. Either W or r may be assumed, and the other computed.

It is sometimes impossible on account of the construction of the rotating part to place the balancing weight in the plane of rotation of the body to be balanced. In this case, it is necessary to use two balancing weights in different planes of rotation. Figure 392(a) illustrates the case in which the balancing weights are in planes of rotation on opposite sides of the plane of rotation of the body to be balanced, and Fig. 392(b) illustrates the case in which

the balancing weights are both on the same side of the plane of rotation of the body to be balanced.

In either figure, let W_1 be the body to be balanced, and B and C the points on the axis in the planes of rotation of the balancing weights. It is necessary that the balancing weights W_2 and W_3 lie in the plane through W_1 and the axis OO as shown. It is also necessary that the sum of the moments of the normal effective forces about any point in this plane shall be equal to zero.

Equation $\Sigma M_B = 0$ gives

$$\frac{W_3}{g}r_3\omega^2 b = \frac{W_1}{g}r_1\omega^2 a$$

or

$$W_3 r_3 b = W_1 r_1 a$$

In Fig. 392(a), equation $\Sigma M_C = 0$ gives

$$W_2 r_2 b = W_1 r_1 (b - a)$$

In Fig. 392(b), the same equation gives

$$W_2 r_2 b = W_1 r_1 (b + a)$$

From these equations the two unknown quantities $W_2 r_2$ and $W_3 r_3$ may be determined.

Similarly, any other bodies W_1', W_1'', etc., in any other planes normal to the axis may be balanced by bodies W_2', W_2'', etc., in the plane through B, and bodies W_3', W_3'', etc., in the plane through C. Then, finally, all the bodies W_2, W_2', W_2'', etc., in the normal plane through B may be replaced by a single body, and the bodies W_3, W_3', W_3'', etc., in the normal plane through C may be replaced by a single body.

Figure 393 represents a slender rod AB at an angle θ with the axis DE about which it rotates, held by the arms DA and EB. Its center of gravity C is in the axis; but if each half, AC and CB, is considered separately, the effective forces reversed form a couple. If l is the length AB, W_1 the weight of the rod, $\bar{r}$ the distance of the center of gravity of each half of the rod from the axis DE, and θ the angle the rod makes with the axis of rotation, the moment of the couple is

$$\frac{W_1}{2g}\bar{r}\omega^2 \times \frac{2}{3}l \cos\theta$$

Since $\bar{r} = \frac{l}{4} \sin \theta$, this reduces to

$$\frac{W_1}{12g} l^2\omega^2 \sin \theta \cos \theta$$

This couple may be balanced by the couple of the reversed effective forces on weights W, W in the planes through D and E, each at a distance r from the axis.

$$\frac{Wr}{g}\omega^2 l \cos \theta = \frac{W_1}{12g} l^2\omega^2 \sin \theta \cos \theta$$

$$Wr = \frac{W_1 l \sin \theta}{12}$$

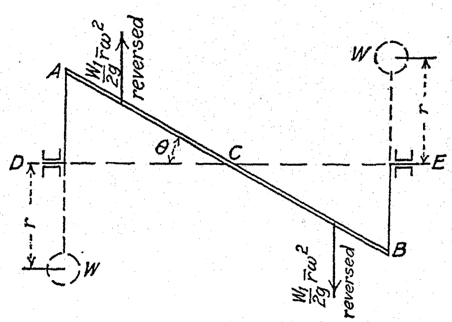


FIG. 393.

Either W or r may be assumed, and the other computed. For perfect balance of the entire rotating body, the arms DA and BE of the slender rod, and also the arms of the balancing weights W, W must be considered in the computation.

Problems

1. A steel rod ¾ in. in diameter and 20 in. long has a cast-iron sphere 4 in. in diameter at one end and is free to rotate about an axis normal to the rod at the other end. The rotating rod and sphere are to be balanced by a cylindrical steel arm 1 in. in diameter and 6 in. long, at the end of which is a lead sphere. Get the necessary diameter of the lead sphere. (Lead weighs 710 lb./cu. ft.) *Ans.* 4.9 in.

2. In Fig. 391, $W_1 = 12$ lb., $W_2 = 10$ lb., $W_3 = 16$ lb., $r_1 = 20$ in., $r_2 = 18$ in., $r_3 = 16$ in., angle $AOB = 90°$, and angle $BOC = 120°$. If r is to be 10 in., get the necessary weight W to balance the system. Get the angle AOD. *Ans.* 5.51 lb.; 109°25′.

3. A shaft 8 ft. long between bearings carries a steel disk 2 ft. in diameter and 4 in. thick at a point 3 ft. from the left-hand bearing. The disk is

keyed to the shaft and is eccentric 6 in. What balancing weights must be added in planes 1 ft. from the bearings at radial distances of 15 in.?

Ans. 136.72 lb.; 68.36 lb.

4. If on the shaft described in Prob. 3 another similar disk is placed 3 ft. from the right end and 90° back of the first, what must be the balancing weights in order to balance both disks? At what angle with the position of the balancing weights in Prob. 3 must they be placed?

Ans. Left, 152.6 lb. at 26°32′; right, 152.6 lb. at 63°28′.

5. In the rotating assembly shown in Fig. 389, what balancing weights will be required in planes through the outer ends of the rotating cylinders, at a distance of 6 in. from the axis of rotation? *Ans.* 16 lb.

6. In Fig. 393, let the rod AB be a steel rod 4 ft. long and 3 in. in diameter. Let the arms DA and EB be steel rods 2 in. in diameter and 1 ft. long. If the arms of the weights W, W are 1 in. in diameter and 8 in. long, with centers of the weights 10 in. from the axis, what amount of weight W will be required at each place? If these balancing weights are cast-iron cylinders 4 in. long, what must be their diameter? *Ans.* 24.9 lb.; 5.52 in.

GENERAL PROBLEMS ON ROTATION

1. A pulley 3 ft. in diameter rotating at 450 r.p.m. is brought to rest in 125 sec. by the force of friction on its shaft, assumed constant. How many revolutions does it make? What is the tangential acceleration of a point on the rim? What are the normal acceleration and the tangential velocity of a point on the rim at the end of 30 sec.?

Ans. 468.75 rev.; $a_t = -0.565$ ft./sec.^{2}; $a_n = 1924$ ft./sec.^{2}; $v = 53.74$ ft./sec.

2. The drum of a steam hoist for a mine cage is 54 in. in diameter. If the cage is to be lowered at the rate of 20 ft./sec. what is the required r.p.m. of the hoist? *Ans.* 85 r.p.m.

3. If the mine cage in Prob. 2 weighs 800 lb. and the moment of inertia of the drum is 168, during what time may the cage be allowed to drop freely before the given velocity is attained? *Ans.* 1.45 sec.

4. A small wooden beam is deflected 1 in. by a weight of 5 lb. If a weight of 2 lb. is placed on the beam, and the beam is set in vibration, what is the time T of one complete vibration? *Ans.* $T = 0.202$ sec.

5. A weight of 5 lb. is supported by a cantilever-beam spring which is deflected 0.12 ft. below its neutral position. If set in vibration, what will be the frequency n? *Ans.* $n = 2.61$.

6. Get the scale of a spiral spring if a weight of 10 lb. hung from it makes 136 vibrations per minute. *Ans.* 5.25 lb./in.

7. A steel rod 1 in. in diameter and 10 ft. long is supported on a horizontal axis normal to the rod 2 ft. from the end. Locate the center of percussion. What is its period of oscillation if used as a pendulum with small amplitude? *Ans.* 5.78 ft.; $T = 2.66$ sec.

8. If the rod described in Prob. 7 is raised to the horizontal position and then released, what are the normal and tangential reactions at the instant of release? What are the normal and tangential reactions as it passes the

45° position? What are the normal and tangential reactions as it passes the vertical position?

Ans. At 0°, $R_n = 0$; $R_t = 12.8$ lb. At 45°, $R_n = 38.48$ lb.; $R_t = 9.05$ lb. At 90°, $R_n = 54.4$ lb.; $R_t = 0$.

9. If the rod described in Prob. 7 is released from rest in the vertical position with the center of gravity above the axis, what are the normal and tangential components of the reaction when it is 60° from the vertical? When in the horizontal position? When in the vertical position?

Ans. At 60°, $R_n = 0.53$ lb.; $R_t = 11.1$ lb. At 90°, $R_n = 27.7$ lb.; $R_t = 12.8$ lb. At 180°, $R_n = 82.1$ lb.; $R_t = 0$.

10. At what angle does the normal reaction on the rod of Prob. 9 change direction? *Ans.* 59°30′.

11. A cast-iron flywheel weighs 4000 lb. and is cast in two parts. The two sections are held together by 14 steel bolts in all, 4 at each joint at the rim and 6 at the hub. If the center of gravity of each half is 4 ft. from the axis, and the maximum speed is to be 300 r.p.m., compute the necessary diameter of each bolt at the root of the thread for an allowable unit tensile stress of 16,000 lb./in.^{2}. *Ans.* 1.18 in.

12. A solid cast-iron flywheel rim is 6 in. wide, 1 in. thick, and 30 in. outside diameter. If the tension in the arms is neglected, what is the unit tensile stress in the rim when it is rotating at a speed of 1500 r.p.m.? *Ans.* 3500 lb./in.^{2}.

13. A cast-iron flywheel 12 ft. in diameter has a rim 16 in. wide and 4 in. thick. If the ultimate strength of cast iron is 24,000 lb./in.^{2} in tension, what speed will rupture the rim if the tension in the arms is neglected? *Ans.* 815 r.p.m.

14. If a flywheel weighing 10,000 lb. is midway between supports and is rotating at 600 r.p.m., what will be the variation in each reaction due to an eccentricity of 0.04 in.? *Ans.* 4090 lb.

15. The wheel of the Brennan monorail-car gyroscope weighed 1000 lb. and was rotated at 3000 r.p.m. on a horizontal shaft midway between bearings. How much eccentricity would cause each reaction to vary from zero to 1000 lb. with each revolution? *Ans.* 0.00392 in.

16. A flywheel weighing 500 lb. is eccentric on its shaft 0.03 in. and is placed 2 ft. from the left bearing and 6 ft. from the right. The shaft weighs 160 lb., and its center of gravity is at the middle. What are the maximum and minimum reactions at each end when the wheel is rotating at 1200 r.p.m.? *Ans.* At left, 915 lb.; −5 lb. At right, 358 lb.; 52 lb.

17. A 20-lb. governor ball with its center 2 ft. from the point of support on the axis is rotating at such a speed that its arm is kept at an angle of 75° with the axis. Compute its speed and the tension in the arm. *Ans.* 75.5 r.p.m.; 77.3 lb.

18. What superelevation is required on a 4° railway curve for zero resultant flange pressure on a car at a speed of 40 m.p.h.? What is the resultant flange pressure on a 100,000-lb. car moving around this curve at a speed of 60 m.p.h.? Gage G of the track is 4.9 ft. *Ans.* $e = 4.36$ in.; 9300 lb.

19. With a radius of 300 ft., at what angle should an automobile speedway be banked for zero lateral pressure on a car at a speed of 60 m.p.h.? If the

tread of a racing car is 4.5 ft., distance of its center of gravity above the ground 1.5 ft., and coefficient of friction $f = 0.6$, what is the limiting speed on the curve with this banking? *Ans.* 38°40′; 110 m.p.h.

20. A vertical shaft 8 ft. long carries a weight of 120 lb. eccentric 3 in. at a point 1 ft. from the upper support, and a weight of 80 lb. eccentric 7.5 in. at a point 6 ft. from the upper support on the opposite side of the shaft. Solve for the normal reactions at the supports when the shaft is rotating at 180 r.p.m. *Ans.* 371 lb. at bottom; 148 lb. at top.

21. Solve for the tangential reactions on the shaft described in Prob. 20 if it is brought to rest in 10 sec. with uniform negative acceleration.
Ans. 1.98 lb. at bottom; 0.80 lb. at top.

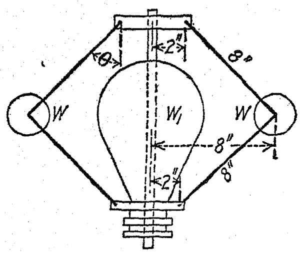


FIG. 394.

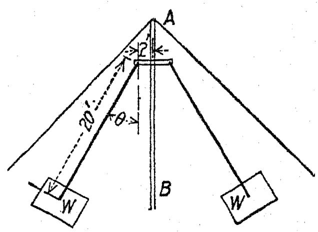


FIG. 395.

22. Figure 394 represents a weighted conical pendulum governor for which $W_1 = 60$ lb. and $W = 15$ lb. What speed will keep the governor in the position shown? What is the tension in each arm?
Ans. 158 r.p.m.; 45.4 lb. in lower; 68.0 lb. in upper.

23. If $\theta = 15°$ when the governor shown in Fig. 394 is in its lowest position, at what speed will it begin to act? *Ans.* 107.6 r.p.m.

24. In the swing shown in Fig. 395 each car weighs 1500 lb. As the cars are rotated about the vertical axis AB, they swing out from the vertical. If the maximum allowable value of θ is 50°, what is the maximum speed at which it may be run? What is the corresponding stress in the supporting cables? *Ans.* 14.25 r.p.m.; 2330 lb.

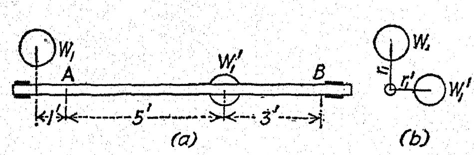


FIG. 396.

25. Figure 396(a) shows a side view, and Fig. 396(b) an end view of a shaft to which weights W_1 and W_1' are attached. If $W_1 = 120$ lb., $r_1 = 18$ in., $W_1' = 80$ lb., and $r_1' = 15$ in., what weights at A and B with radii of 12 in. will be necessary to balance the system? What is the angle of each with the horizontal plane?

Ans. $W_A = 206$ lb.; $\theta_A = 259°30'$; $W_B = 66.4$ lb.; $\theta_B = 160°10'$.

26. A motorcycle travels horizontally around on the inside of a hollow vertical cylinder 100 ft. in diameter. What is the minimum coefficient of friction that will permit this to be done at a speed of 40 m.p.h.?

Ans. $f = 0.468$.

27. A helical spring is suspended at its upper end and supports a weight at its lower end. If the weight is pulled down 1.5 in. and then suddenly released, it makes a complete vibration in 0.72 sec. Get its displacement from its static position, its velocity, and its acceleration 3 sec. after it is released. *Ans.* -0.75 in.; 0.945 ft./sec.; 4.76 ft./sec.^{2}.

28. If the weight referred to in Prob. 27 is pulled down 2 in. and then suddenly released, solve for the time after release when the displacement is 1 in., the time after release when the velocity is -0.5 ft./sec., and the time after release when the acceleration is -2 ft./sec.^{2}.

Ans. 0.24 sec.; 0.40 sec.; 0.198 sec.

29. The rotor of a gyroscope is 5 ft. in diameter and is brought from rest up to a speed of 4500 r.p.m. in 5 min. Compute the angular acceleration, the tangential acceleration of a point on the rim, and the normal acceleration of a point on the rim when full speed is attained.

Ans. 1.571 rad./sec.^{2}; 3.93 ft./sec.^{2}; 555,180 ft./sec.^{2}.

30. The balance wheel of a watch is 0.6 in. in diameter and makes a complete vibration in $\frac{1}{2}$ sec. The angular displacement is 15° each way from the neutral position. The angular acceleration is assumed to vary inversely as the angle of displacement. Compute the maximum angular velocity, the maximum linear velocity of a point on the rim, the maximum angular acceleration, and the maximum normal acceleration of a point on the rim.

Ans. 3.29 rad./sec.; 0.0823 ft./sec.; 41.36 rad./sec.^{2}; 0.271 ft./sec.^{2}.

31. In a diving test, an airplane had a vertical downward speed of 579 m.p.h. and was then pulled out into the horizontal in a curve with a radius short enough at one place so that an acceleration of 7.6g was caused. Compute the required radius. *Ans.* $r = 2950$ ft.

CHAPTER XV

ANY PLANE MOTION OF RIGID BODIES

143. Combined Translation and Rotation.—In a plane motion of a body, each point of the body remains at a constant distance from a fixed plane. This plane or any parallel plane may be called the *plane of motion* of the body. The plane of motion through the center of gravity of the body is commonly used for reference.

Any plane displacement of a body may be considered to be made up of a rotation about any point in the plane of motion and a corresponding translation. That is, the same result would have

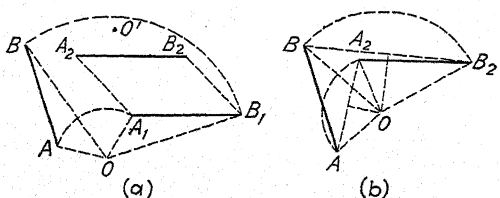


FIG. 397.

been obtained by the two simple motions as by the actual motion, whatever it may have been. In Fig. 397(a) let AB be a line in the plane of motion joining any two points of a body in their original position, and let A_2B_2 be their position after any plane motion of the body. Let O be any point in the plane of motion. The displacement from AB to A_2B_2 may evidently be made by a rotation about O to the position A_1B_1 parallel to A_2B_2, then a translation from A_1B_1 to A_2B_2.

Again, any plane displacement of a body is equivalent to a simple rotation about some fixed point in space. In order to locate this point, join AA_2 and BB_2, Fig. 397(b). Erect perpendicular bisectors of AA_2 and BB_2 which intersect at O. The triangles AOB and A_2OB_2 are equal in all their parts. Therefore angle AOA_2 = angle BOB_2, since angle AOB = angle A_2OB_2. Hence it is plain that the displacement from AB to A_2B_2 is equivalent to simple rotation through angle AOA_2 about point O.

If AA_2 and BB_2 are parallel, the point Q is at infinity and the motion is equivalent to pure translation.

Problems

1. In Fig. 397(*a*), resolve the displacement from AB to A_2B_2 into a rotation about O' and a corresponding translation. Do the same, using point A as a center. Do the same, using the point midway between A and B as a center.

2. A wheel rolls along a horizontal plane through ¼ revolution. Find the center of equivalent rotation. Do the same for rotation through ½ revolution.

144. Composition and Resolution of Velocities in Any Plane Motion.—The absolute velocity of any point of a body that has any plane motion can usually be determined best by getting first its velocity relative to some special point of reference on the body

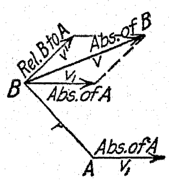


FIG. 398.

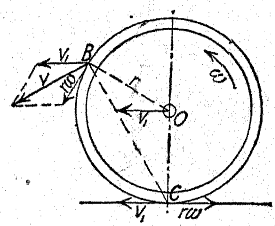


FIG. 399.

and then the absolute velocity of the point of reference. By Art. 101 the absolute velocity of the given point is equal to the vector sum of the two velocities, its own velocity relative to some point of reference, and the absolute velocity of the point of reference.

Let B, Fig. 398, be any point of a rigid body and let A be the point of reference chosen. Let v_1 be the absolute velocity of A and v' the velocity of B relative to A. Since A and B are fixed points on the rigid body, the only velocity B can have relative to A is tangential. This is equal to $r\omega$, r being the length AB and ω the angular velocity. By the principle stated above, the absolute velocity of B is the vector sum of the two, or v.

Conversely, the absolute velocity of any point of a rigid body may be resolved into two components, one of which is equal and parallel to the absolute velocity of any chosen point of reference on the body, and the other is normal to the line joining the two points.

As an example, consider the wheel shown in Fig. 399 which is rolling to the left on a horizontal plane. If the velocity of the

heel is v_1 and its angular velocity is ω, the velocity
the rim with respect to the center is $r\omega = v_1$ for
e absolute velocity of any point is the vector sum
th respect to the center and the absolute velocity
shown at B. The absolute velocity of B is v, the
and $r\omega$. In the same way, the absolute velocity
ANY Pl nt C is the vector sum of its velocity with respect
143. Combined and the absolute velocity of O, which is v_1.
of a body, each pc ual in amount and opposite in direction, the
om a fixed plar f point C is zero.
d the *plane* e motion of the rolling wheel, it is sometimes
simp gh the cent it C as the point of reference. Since its abso-
lute v ce.), the absolute velocity of any point, as B, is the
same a plane dis y relative to point C. Then $v_B = r_B\omega$, r_B being
the dist- of a rota
nding tr

Problems

1. In Fig. 60 ft./sec. horizontal to the right, r = 6 ft., ω = 12
d./sec. cloc. A_2 d AB makes an angle of 30° with the horizontal.
lve for the ab elocity of B. *Ans.* v = 114.5 ft./sec., 33° with H.

2. A cylinder ft. diameter is rolling to the right on a horizontal plane
h a uniform vel y of 12 ft./sec. Using the center as the point of
reference find the a ute velocity of a point on the rim in front, 30° above
the hori tal throu the center. Check by using the bottom point of
the cylinder as the point of reference. *Ans.* 20.8 ft./sec., at 30° with H.

3. The propeller of an airplane is geared to make 11 revolutions to 16 revolutions of the engine. With the plane at a speed of 302 m.p.h., and the engine rotating at 2300 r.p.m., what is the speed relative to the air of the tip of the propeller blade 11 ft. in diameter? *Ans.* 1013 ft./sec.

145. Composition and Resolution of Accelerations in Any Plane Motion.—The principle of Art. 101, which was referred to in the preceding article, is true for accelerations as well as for velocities. That is, the absolute acceleration of any given point of a body is equal to the vector sum of the relative acceleration of the given point with respect to a chosen point of reference on the body and the absolute acceleration of the point of reference.

Let B, Fig. 400, be any point of a rigid body and let A be the point of reference whose absolute acceleration is a_1. Let $AB = r$ and let the angular velocity and acceleration be ω and α, respectively. The relative acceleration is most easily determined by means of its tangential and normal components. The tangential component is $r\alpha$ and the normal component is $r\omega^2$. These two

combined give the relative acceleration a'; and fir ıty and combined give vector $BC = a$, the absolute ac

Conversely, the absolute acceleration of an resolved into three components, one of which is e to the absolute acceleration of any chosen po ' to A_2B_2 into a on the body, another equal to $r\omega^2$ along the o the same, using two points, and a third equal to $r\alpha$ perpendicul y between A and B

As an example, consider the wheel shown in ¼ revolution. Find rolling to the left on a horizontal plane. Let or rotation through the center of the wheel be a_1, its angular acce angular velocity ω. The tangential compone **ies in Any Pla-**

a body tha
t by gettir
ence on th

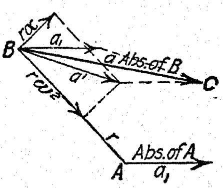
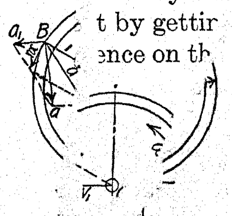


FIG. 400.



tion of any point on the rim, as B, relative t ne center is $r\alpha$ a_1 for free rolling. The normal compone of the relativ acceleration is $r\omega^2$. These two vectors con ined giv a', th relative acceleration of B with respect to the center. The vector sum of a' and a_1 gives a, the absolute acceleration of point B.

Problems

1. In Fig. 400, $a_1 = 30$ ft./sec.² horizontal to the right; AB is 6 ft. long, at an angle of 30° with the horizontal; $\omega = 6$ rad./sec. clockwise, and $\alpha = 10$ rad./sec.² clockwise. Get the absolute acceleration of point B.

Ans. $a = 253.2$ ft./sec.², 12°47′ below H.

2. Solve for the absolute acceleration of the point referred to in Prob. 2, Art. 144, using the same two points of reference.

Ans. 72 ft./sec.² toward the center.

3. In Fig. 401, let the wheel be 6 ft. in diameter, and let it be rolling to the left with $\omega = 8$ rad./sec., but let $\alpha = -12$ rad./sec.². Solve for the absolute acceleration a of point B, if radius OB is 30° with the horizontal.

Ans. 229.4 ft./sec.², 16°25′ below H.

146. Instantaneous Axis of Rotation.—The *instantaneous center* of a rigid body having any plane motion is the center about which rotation is taking place at the instant considered. If the instantaneous center is in the rigid body, that point on the

body is at rest at that instant. The *instantaneous axis* is the axis through the instantaneous center normal to the plane of motion.

To locate the instantaneous center, it is necessary to know the direction of the velocity of at least two points of the body, the two points being on different radii. Let v_A, Fig. 402, be the velocity of point A, and v_B the velocity of point B on the rigid body. Line AC is drawn normal to the direction of the velocity v_A. Then if there is a center of rotation of the rigid body, it must be at some point on the line AC. Similarly, line BD is drawn normal to the direction of the velocity v_B. Then if there is a center of rotation of the rigid body, it must be at a point on line BD. If lines AC and BD intersect, the instantaneous center is at their point of intersection, O in this case. If lines AC and BD do not intersect, v_A and v_B are parallel and the motion of the body at that instant is pure translation.

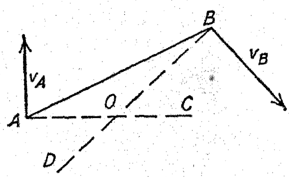


FIG. 402.

In general, the instantaneous center will be at a different point the following instant, both in space and relative to the body. Its path relative to the body is called the *body centrode* and its path in space is called the *space centrode.*

In instantaneous rotation, as in simple rotation, the relation $v = r\omega$ holds true. Then $v_B = \overline{OB}\omega$, and $v_A = \overline{OA}\omega$, ω being the angular velocity of the body at that instant.

Problems

1. Locate the instantaneous center of the connecting rod AB, Fig. 403, in the position shown. What is the velocity of the crosshead A if the crankpin B is rotating at 100 r.p.m.?

Ans. 11.15 ft. above A; $v_A = 9.85$ ft./sec.

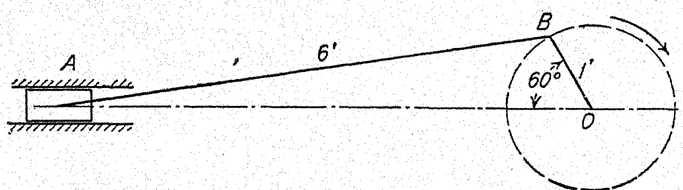


FIG. 403.

2. The center of gravity of the connecting rod shown in Fig. 403 is 4 ft. from A. Get the velocity and the normal acceleration of the center of gravity in the position shown.

Ans. $\bar{v} = 9.95$ ft./sec., 20°30′ with H; $a_n = 8.78$ ft./sec.^{2}.

3. In the link motion shown in Fig. 404, OA is rotating at 6 rad./sec. in the clockwise direction. Locate the instantaneous center of link AB. Compute the velocity of point B and the angular velocity of QB.

Ans. 18.9 ft. from A on OA produced; v_B = 12.3 ft./sec.; ω of QB = 3.075 rad./sec.

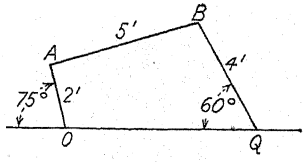


FIG. 404.

147. General Equations of Motion. As shown in Art. 145, the absolute acceleration of any point of a rigid body having any plane motion may be resolved into three components, one equal and parallel to the acceleration of any chosen point of reference on the body, another equal to $\rho\omega^2$ acting along the line joining the two points, and a third equal to $\rho\alpha$ acting perpendicular to this line. If the mass of any particle is dM, the effective force for it is equal to the resultant of the three components dMa, $dM\rho\omega^2$, and $dM\rho\alpha$.

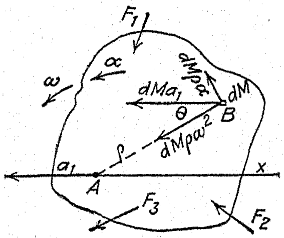


FIG. 405.

In Fig. 405 let A be the point of reference, and let the X axis coincide with a_1, the absolute acceleration of A. Let ΣF be the resultant of all the external forces, and ΣF_x and ΣF_y the components of ΣF in the X and Y directions, respectively. Since the external force system is equivalent to the effective force system for the whole body of mass M,

$$\begin{aligned}
\Sigma F_x &= \int dMa_1 + \int dM\rho\alpha \sin\theta + \int dM\rho\omega^2 \cos\theta \\
&= a_1\int dM + \alpha\int dMy + \omega^2\int dMx \\
&= Ma_1 + M\alpha\bar{y} + M\omega^2\bar{x} \\
\Sigma F_y &= \int dM\rho\alpha \cos\theta - \int dM\rho\omega^2 \sin\theta \\
&= \alpha\int dMx - \omega^2\int dMy \\
&= M\alpha\bar{x} - M\omega^2\bar{y} \\
\Sigma M_A &= \int dM\rho^2\alpha + \int dMa_1\rho \sin\theta \\
&= \alpha\int \rho^2\, dM + a_1\int dMy \\
&= I_A\alpha + Ma_1\bar{y}
\end{aligned}$$

If the center of gravity is taken as the point of reference, as is usually the case, the three equations above become

$$\Sigma \mathbf{F}_x = \mathbf{M}\bar{\mathbf{a}}$$
$$\Sigma \mathbf{F}_y = \mathbf{0}$$
$$\Sigma \mathbf{M}_O = \mathbf{I}_O\alpha$$

I_O is the moment of inertia of the body with respect to the axis through the center of gravity.

The principles derived above may be stated as follows:

I. In any plane motion the center of gravity is accelerated the same as if the whole mass were concentrated at that point and acted upon by forces equal in amount and direction to the actual forces.

II. In any plane motion the angular acceleration about the center of gravity is the same as if that center were fixed and a couple of moment $= \Sigma M_O$ applied to the body.

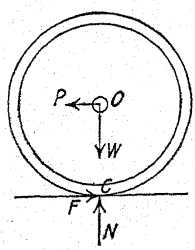


FIG. 406

If the reversed effective forces are considered to be added to the free body with its actual impressed forces, the equations of equilibrium hold true. That is, if a force $M\bar{a}$ is applied at the center of gravity in a direction opposite to its absolute acceleration, and a couple $I_O\alpha$ opposed in direction to the angular acceleration, the problem will be reduced to static conditions.

148. Rolling Wheel.—The motion of a wheel rolling upon a plane is a motion of combined translation and rotation. In Fig. 406 the horizontal force P acts at the center of the wheel to produce free rolling of the wheel over the plane. Since the wheel tends to slide over the plane surface, a frictional force F is induced at the point of contact, opposing the motion. The center of gravity O may be taken as the point of reference, so the special equations of motion given in Art. 147 may be used

$$\Sigma F_x = P - F = \frac{W}{g}a$$
$$\Sigma F_y = W - N = 0$$
$$\Sigma M_O = Fr = I\alpha$$

For free rolling,

$$a = r\alpha$$

From these four equations the acceleration and the frictional force may be determined.

If a wheel is rolling on an inclined plane, a component of the weight of the wheel will be acting parallel to the plane and so will influence the motion. Let Fig. 407 represent a wheel rolling on a plane inclined at an angle β with the horizontal. The plane is considered to be rough enough so that free rolling takes place. If the wheel is released from rest at the top of the plane and allowed to roll down freely under the influence of gravity, there will be acting, in addition to W, the normal reaction N and the frictional force F. Again, the center of gravity O may be taken as the point of reference, and the special equations of Art. 147 used. The X axis must be taken in the direction of the acceleration of the point of reference, parallel to the plane.

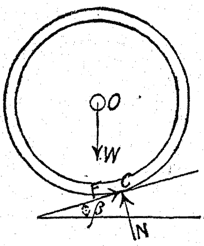


FIG. 407.

$$\Sigma F_x = W \sin \beta - F = \frac{W}{g}a$$
$$\Sigma F_y = W \cos \beta - N = 0$$
$$Fr = I_O \alpha$$

Also, $a = r\alpha$ for free rolling.

EXAMPLE 1

Let the wheel shown in Fig. 406 be a solid circular cylinder 4 ft. in diameter weighing 1000 lb., and let $P = 100$ lb. Solve for F, a, and α. Solve for the velocity of the cylinder after it has moved 5 ft. from rest.

Solution.—The equation $\Sigma F_x = Ma_x$ gives

$$100 - F = \frac{1000}{32.2}a$$

The equation $\Sigma M_O = I_O\alpha$ gives

$$2F = \frac{1}{2} \times \frac{1000}{32.2}4\alpha$$

For free rolling,

$$a = 2\alpha$$

The solution of these three equations gives

$$F = 33.33 \text{ lb.}$$
$$a = 2.147 \text{ ft./sec.}^2$$
$$\alpha = 1.073 \text{ rad./sec.}^2$$

The equation $v^2 = v_0^2 + 2as$ gives

$$v^2 = 21.47$$
$$v = 4.633 \text{ ft./sec.}$$

EXAMPLE 2

Figure 408 represents a cylinder 2 ft. in diameter weighing 200 lb. resting on a horizontal plane surface. Attached to the cylinder and concentric with it is a hollow cylinder 1 ft. in diameter, of negligible weight, around which a cord is wrapped. What horizontal force P applied to the cord as shown will produce an acceleration of 20 ft./sec.^{2}? What is the frictional force F? Assume free rolling. What is the linear velocity of the center after moving 8 ft. from rest?

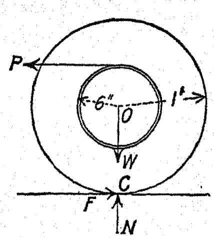


FIG. 408.

Solution.—Equation $\Sigma F_x = Ma_x$ gives

$$P - F = \frac{200}{32.2} \times 20$$

Equation $\Sigma M_O = I_O\alpha$ gives

$$P \times 0.5 + F \times 1 = \frac{1}{2} \times \frac{200}{32.2} \times 1^2 \times \frac{20}{1}$$
$$P = 124.2 \text{ lb.}$$
$$F = 0$$

If P is applied lower than the point indicated, a frictional force is induced in the direction opposite to P, whereas, if it is applied above this point, a frictional force is induced in the same direction.

$$v^2 = 2as = 320$$
$$v = 17.89 \text{ ft./sec.}$$

EXAMPLE 3

Prove that $f = \frac{2}{7} \tan \beta$ is the minimum value of f for free rolling of a solid homogeneous sphere on a plane at an angle of β with the horizontal.

Solution.

$$I_O = \frac{2}{5}\frac{W}{g}r^2$$

The equation $\Sigma M_O = I\alpha$ gives

$$Fr = \frac{2}{5}\frac{W}{g}r^2\alpha$$
$$F = \frac{2}{5}\frac{W}{g}a$$

The equation $\Sigma F_x = \frac{W}{g}a$ gives

$$W \sin \beta - F = \frac{W}{g}a$$

By eliminating $\frac{W}{g}a$ between these two equations,

$$F = \frac{2}{7}W \sin \beta$$

As angle β is increased, the value of $\sin \beta$ increases until F reaches its limiting value F', and slipping impends. For this value of β,

$$f = \frac{F'}{N} = \frac{\frac{2}{7} W \sin \beta}{W \cos \beta}$$
$$f = \tfrac{2}{7} \tan \beta$$

If $\frac{2}{7} \tan \beta$ is larger than the static coefficient of friction f, the sphere will slip at the point of contact, and a will not be equal to $r\alpha$. If the coefficient of kinetic friction is known, however, the force of friction becomes known, and all values may be determined.

EXAMPLE 4

A wheel 2 ft. in diameter weighing 100 lb. starts from rest and rolls freely down a 30° incline 10 ft. long. The radius of gyration of the wheel is 0.8 ft. Get the linear acceleration and the linear velocity of the center of the wheel at the foot of the incline. Get the frictional force F.

Solution.—The center of the wheel may be used as the point of reference, so the special equations of motion apply. The X axis is parallel to the plane. The equation $\Sigma F_x = Ma$ gives

$$50 - F = \frac{100}{32.2} a$$

The equation $\Sigma M_O = I_O \alpha$ gives

$$F = \frac{100}{32.2} \times 0.64$$

Since $r = 1$ ft., $a = \alpha$ for free rolling. If a is substituted for α, and the two foregoing equations are added,

$$\begin{aligned} 50 &= \frac{164}{32.2} a \\ a &= 9.82 \text{ ft./sec.}^2 \\ F &= 19.5 \text{ lb.} \\ v^2 &= 2as = 2 \times 9.82 \times 10 = 196.4 \\ v &= 14.02 \text{ ft./sec.} \end{aligned}$$

Problems

1. A cast-iron cylinder 1 ft. in diameter and 1 ft. long, free to roll on a horizontal plane, has a force of 16 lb. acting horizontally at the center normal to the geometric axis. Find the velocity and the acceleration of a point on the rim behind, 45° above the horizontal through the center, 5 sec. from rest. Solve also for the frictional force F.

Ans. $v = 9$ ft./sec. forward and upward, 22°30′ with H; $a = 48.2$ ft./sec.^{2} forward and downward, 43° with H; $F = 5.33$ lb. backward.

2. Solve Prob. 1 if the 16-lb. force is acting horizontally at the top of the cylinder.

Ans. $v = 18$ ft./sec. forward and upward, 22°30′ with H; $a = 192$ ft./sec.^{2} forward and downward, 44° with H; $F = 5.33$ lb. forward.

3. In Fig. 408, let the radius of the small hollow cylinder be 9 in., and the value of P be 50 lb. Solve for the acceleration a and the frictional force F.

Ans. $a = 9.39$ ft./sec.^{2}; $F = 8.33$ lb. forward.

4. In Fig. 408, let the small cylinder be solid, 6 in. in diameter, and weigh 80 lb. Solve for the force P to give an acceleration of 8 ft./sec.^{2}. Solve also for the frictional force F. *Ans.* $P = 76$ lb.; $F = 6.5$ lb. backward.

5. Prove that $f = \frac{1}{3} \tan \beta$ is the minimum value of f for free rolling of a solid homogeneous circular cylinder on a plane at an angle β with the horizontal.

6. A cast-iron sphere 6 in. in diameter and a cast-iron cylinder also 6 in. in diameter and weighing the same as the sphere are released from rest at the top of a 15° plane 40 ft. long and allowed to roll down freely. Get the time required for each to reach the bottom of the plane and also the frictional force F under each.

Ans. Sphere, $t = 3.67$ sec., $F = 2.18$ lb.; cylinder, $t = 3.8$ sec., $F = 2.54$ lb.

7. If the cylinder shown in Fig. 408 is placed on a 30° plane, get the amount of force P parallel to the plane that will give the center of the cylinder an acceleration of 4 ft./sec.^{2} up the plane. Get also the amount of the frictional force F. Assume free rolling.

Ans. $P = 91.52$ lb.; $F = 33.33$ lb. up the plane.

149. Kinetic Reaction on Unbalanced Rolling Wheel.—If a wheel whose center of gravity does not coincide with its geometric center rolls along a horizontal plane surface, the reaction of the surface is not constant in amount but changes during each revolution from a value greater than the weight W to one less than W. Let the wheel be as shown in Fig. 409, with its center at O and its center of gravity at C, due to the added weight on one side. Let the speed of the center be constant toward the left and let its amount be v. Since the pull of gravity tends to retard the motion for values of θ from 0° to 180° and to accelerate it for values of θ from 180° to 360°, the forces F_1 and F are necessarily variable.

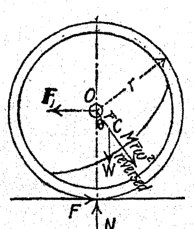


FIG. 409.

By the principle of relative motion, the absolute acceleration of point C is equal to the vector sum of the absolute acceleration of the center $r\alpha$ and the relative acceleration of C with respect to the center. The two components of the relative acceleration are $\bar{r}\alpha$ and $\bar{r}\omega^2$. Since α is zero, the acceleration of point C is $\bar{r}\omega^2$ toward the center O. If, now, the reversed effective force $M\bar{r}\omega^2$ is added, the wheel will be under static conditions and the equations of equilibrium may be written. Equation $\Sigma F_y = 0$ gives

$$N = W + M\bar{r}\omega^2 \cos \theta$$

When θ is zero, the value of N is a maximum; and when θ is 180°, it is a minimum.

Problems

1. In Fig. 409, let $r = 2$ ft., and $\bar{r} = 3$ in. At what speed will the reaction N be zero when C is directly above O? *Ans.* 22.72 ft./sec.

2. A locomotive driving wheel 6 ft. in diameter weighs 1800 lb. and carries an axle load of 12,000 lb. When the side rod and the connecting rod are removed, the center of gravity of the wheel is 0.6 ft. from the center of the wheel because of the counterweight. If the locomotive is pulled by another at a speed of 40 m.p.h., what is the variation in the pressure on the track during one revolution? *Ans.* 26,620 to 980 lb.

150. Connecting Rod of Engine. *Graphic Solution.*—The principle of relative velocities and accelerations is especially well

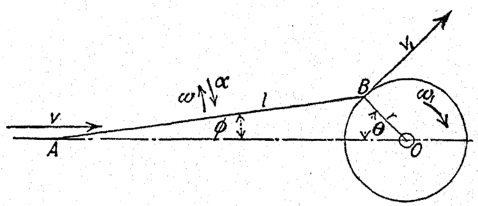


FIG. 410.

adapted to the solution of the problem of the connecting rod of a steam engine. In Fig. 410, A is the crosshead with velocity v and acceleration a, these being the same as the velocity and acceleration of the piston; AB is the connecting rod, of length l; B is the crankpin with tangential velocity v_1; BO is the crank of length r; O is the center of rotation of the flywheel. The flywheel is assumed to be heavy enough so that point B has a rotation practically uniform, with angular velocity ω_1. The angle ϕ between the connecting rod and the line AO increases to a maximum when B is at the top point in its circle, then decreases and changes to negative values.

The motion of the connecting rod is an oscillatory rotation with variable angular velocity ω about point A, which meanwhile has an oscillatory translation along the line AO. The absolute velocity of point B is $v_1 = r\omega_1$ normal to OB. By the principle of relative velocities, this is equal to the vector sum of the absolute velocity of A and the relative velocity of B with respect to A. Let v_2 be the velocity of B relative to A. It is known that its direction is normal to AB, since A and B are rigidly connected, and that the direction of v, the velocity of point A, is horizontal.

The vector diagram (Fig. 411) completely determines the value of v_2 and v. Since the motion of B relative to A is a rotation with radius l, $v_2 = l\omega$, or $\omega = v_2/l$. The linear velocity of A and the angular velocity of the rod with respect to A are thus completely determined, so the absolute velocity of any point on the rod may be found.

In Fig. 412 the unknown accelerations are determined. Since point B, Fig. 410, is moving in a circle with uniform angular velocity ω_1, its only acceleration is $r\omega_1^2$, toward the center O. This is drawn to scale in Fig. 412. The relative acceleration of B with respect to A and the absolute acceleration of A must have $r\omega_1^2$ as their vector sum. The relative acceleration of B with respect to A is made up of two components, one wholly known, the other known only in direction. The normal component $l\omega^2$ is completely known, so is drawn first. The tangential component $l\alpha$ is perpendicular to the direction of the connecting rod, and the absolute acceleration a of point A is horizontal and must close the polygon. This completely determines a and α, so the absolute acceleration of any point on the rod may be found.

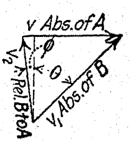


FIG. 411.

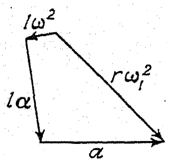


FIG. 412.

Problems

1. The connecting rod of an engine is 6 ft. long, and the crank is 1 ft. long. If the crank is rotating at 180 r.p.m., get the velocity and the acceleration of the crosshead when $\theta = 0°$ and when $\theta = 45°$.

Ans. For $\theta = 0°$, $v = 0$; $a = 414.5$ ft./sec.^{2}. For $\theta = 45°$, $v = 14.9$ ft./sec.; $a = 251.4$ ft./sec.^{2}.

2. Solve for the velocity and the acceleration of the crosshead of the engine described in Prob. 1 when $\theta = 90°$, when $\theta = 120°$, and when $\theta = 180°$.

Ans. For $\theta = 90°$, $v = 18.85$ ft./sec.; $a = -60$ ft./sec.^{2}. For $\theta = 120°$, $v = 14.95$ ft./sec.; $a = -207.2$ ft./sec.^{2}. For $\theta = 180°$, $v = 0$; $a = -296.1$ ft./sec.^{2}.

151. Kinetic Reactions on Connecting Rod.—In order to determine the crankpin and crosshead-pin pressures, the connecting rod is considered as a free body (Fig. 413). The impressed forces acting upon the rod consist of the following: its weight W vertically downward at its center of gravity; the pressure F from the piston rod through the crosshead; the normal pressure

N_A from the crosshead; the crankpin reaction at B. The crankpin reaction is resolved into its two components, N along the rod and T perpendicular to the rod. Of these impressed forces, N, T, and N_A are unknown.

Under the action of these impressed forces the rod is accelerated both in translation and in rotation. By the method of Art. 150 the values of ω, α, and a may be determined. If, now, the reversed effective forces are added to the free body (Fig. 413), it

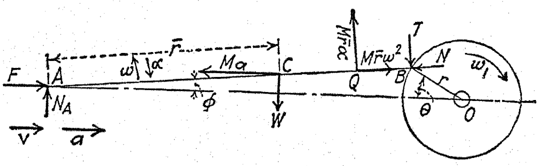


FIG. 413.

will be under static conditions as discussed in Art. 147. If ω_1 is clockwise and the value of θ is between 0° and 90°, ω, α, v, and a will be in the directions shown. The three components of the acceleration of the center of gravity are a horizontal to the right; $\bar{r}\alpha$ downward, normal to the rod; and $\bar{r}\omega^2$ along the rod toward A.

Then the *reversed effective* forces are (1) Ma horizontal to the left; (2) $M\bar{r}\omega^2$ outward away from A; and (3) $M\bar{r}\alpha$ upward, normal to the rod. It will be remembered that $M\bar{r}\alpha$ acts through the point Q, distant $k^2/\bar{r}$ from A, and not through the center of gravity. The force $M\bar{r}\omega^2$ acts through the center of gravity as shown in Case 1, Art. 132. Since all the forces are known except N_A, N, and T, these may be determined by the solution of the three equations of equilibrium.

In Fig. 413, equation $\Sigma F_x = 0$ gives

$$F - \mathrm{N}\cos\phi + T\sin\phi - Ma + M\bar{r}\omega^2\cos\phi - M\bar{y}\alpha = 0$$

Equation $\Sigma F_y = 0$ gives

$$N_A - W + M\bar{x}\alpha + M\bar{r}\omega^2\sin\phi - T\cos\phi - N\sin\phi = 0$$

Equation $\Sigma M_A = 0$ gives

$$W\bar{x} - Ma\bar{y} - M\bar{r}\alpha\left(\frac{k^2}{\bar{r}}\right) + Tl = 0$$

EXAMPLE

In Fig. 413, let $r = 1$ ft., $l = 6$ ft., $W = 200$ lb., $\bar{r} = 3.8$ ft., $F = 10{,}000$ lb., $\omega_1 = 30$ rad./sec., $\theta = 30°$, and $I_A = 120$. Solve for N_A, N, and T.

Solution.

$$M = \frac{W}{g} = \frac{200}{32.2} = 6.21$$
$$\frac{k^2}{\bar{r}} = \frac{I}{M\bar{r}} = \frac{120}{6.21 \times 3.8} = 5.08 \text{ ft.}$$
$$\sin \phi = 0.0833;\ \cos \phi = 0.9965;\ \phi = 4°47'$$
$$\bar{x} = 3.786 \text{ ft.};\ \bar{y} = 0.316 \text{ ft.}$$

The vector diagram for the velocities is shown in Fig. 414, from which, by scale,

$$v = 17.2 \text{ ft./sec.}$$
$$l\omega = 26.1 \text{ ft./sec.}$$
$$\omega = \frac{26.1}{6} = 4.35 \text{ rad./sec.}$$

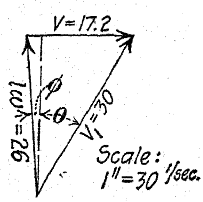


FIG. 414.

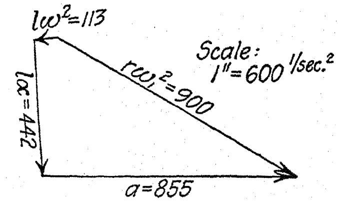


FIG. 415.

In order to determine the accelerations, their vector diagram is drawn (Fig. 415).

$$r\omega_1^2 = 900;\ l\omega^2 = 6 \times 4.35^2 = 113$$

Vector $l\alpha$ scales 442, from which $\alpha = 73.7$ rad./sec.^{2}.
Vector a scales 855 ft./sec.^{2}.

$$Ma = 6.21 \times 855 = 5310 \text{ lb.}$$
$$M\bar{r}\alpha = 6.21 \times 3.8 \times 73.7 = 1740 \text{ lb.}$$
$$M\bar{r}\omega^2 = 6.21 \times 3.8 \times 4.35^2 = 445 \text{ lb.}$$

These are the three components of the effective force for the body, and if applied reversed, as shown in Fig. 413, are in equilibrium with the impressed forces.

Equation $\Sigma F_x = 0$ gives

$$10{,}000 - 5310 - (1740 \times 0.0833) + (445 \times 0.9965) + 0.0833T - 0.9965N = 0$$

Equation $\Sigma F_y = 0$ gives

$$N_A - 200 + (1740 \times 0.9965) + (445 \times 0.0833) - 0.9965T - 0.0833N = 0$$

Equation $\Sigma M_A = 0$ gives

$$(5310 \times 0.316) - (200 \times 3.786) + (1740 \times 5.08) - 6T = 0$$

Solution of the last equation gives

$$T = 1627 \text{ lb.}$$

This value substituted in the first equation gives

$$N = 5150 \text{ lb.}$$

These two values substituted in the second equation give

$$N_A = 477 \text{ lb.}$$

The resultant pressure of the crankpin is given by

$$\sqrt{N^2 + T^2} = 5400 \text{ lb.}$$

Problems

1. The connecting rod described in Prob. 2, Art. 116, is 6 ft. long; the crank is 1 ft. long; the horizontal pressure from the crosshead pin is 6000 lb.; and the engine is running at 180 r.p.m. Get the pressure of the guide on the crosshead and the total pressure on the crankpin when $\theta = 60°$.

Ans. $N_A = 523$ lb.; crankpin pressure = 5010 lb.

2. Solve Prob. 1 for $\theta = 90°$, if $P = 5000$ lb.

Ans. $N_A = 514$ lb.; crankpin pressure = 5610 lb.

152. Kinetic Reactions on Side Rod.—In the side, or parallel, rod of a locomotive, each particle of mass dM, as at D, Fig. 416, has a motion of rotation about its own center, point C on the

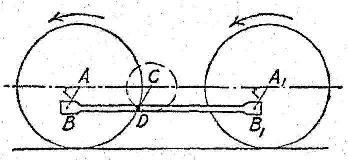


FIG. 416.

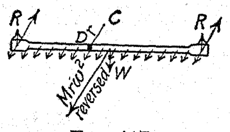


FIG. 417.

line AA_1, and a translation the same as point C. If the linear velocity of the locomotive is constant, the absolute acceleration of point D is $r\omega^2$ directed toward point C, ω being the angular velocity of the wheels. Since at any instant the accelerations of all the particles of the rod are the same in amount and direction, the resultant of all the elementary effective forces is equal to their sum $Mr\omega^2$ and acts through their center of gravity.

Upon the rod as a free body (Fig. 417), the impressed forces acting are the two crankpin pressures and its weight W. The effective force $Mr\omega^2$ if added to the impressed forces reversed in direction will give static conditions. The crankpin pressures are most easily determined in terms of their two components, one vertically upward equal to $W/2$, the other radial equal to $Mr\omega^2/2$. It is evident that the resultant reaction is a maximum when the rod is at the bottom of its travel. At this point,

$$R = \frac{Mr\omega^2}{2} + \frac{W}{2}$$

The reaction is a minimum when the rod is at the top, where

$$R = \frac{Mr\omega^2}{2} - \frac{W}{2}$$

Problems

1. A side rod weighs 450 lb.; the drive wheels are 6 ft. in diameter; the length of the crank is 16 in.; and the locomotive is running at a speed of 75 m.p.h. Get the maximum and minimum pressures on each crankpin.
Ans. 12,750 lb.; 12,300 lb.

2. At what speed will the minimum reaction be zero on the crankpin of the locomotive of Prob. 1? *Ans.* 10.05 m.p.h.

153. Balancing Reciprocating Parts.—A simple illustration of the balancing of reciprocating parts is furnished by the slotted slider apparatus driven by a crank rotating at constant speed, as

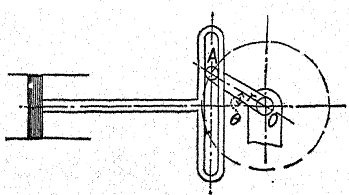


FIG. 418.

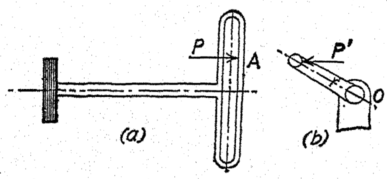


FIG. 419.

shown in Fig. 418. Let W be the weight and $M = W/g$ the mass of the slider, and let friction be neglected. If the angular velocity of the crank AO is ω_1, the acceleration of the crankpin is $r\omega_1^2$ toward the center. The slider has a variable horizontal acceleration $a = r\omega_1^2 \cos\theta$, and its motion is simple harmonic. The force to cause this acceleration is the variable pressure of the crankpin at A, and is equal to

$$P = Ma = Mr\omega_1^2 \cos\theta$$

as shown in Fig. 419(a). For values of θ between 90° and 270°, the acceleration is toward the left, so the force P is acting toward the left on the slider. The equal and opposite pressure of the slider on the crankpin is P', as shown in Fig. 419(b), which is, in turn, transmitted to the support at O.

In order to balance the force P' on the crankpin, a mass M_1 at a distance r_1 may be added opposite the crank, as shown in Fig. 420. If the values of M_1 and r_1 are such that

$$M_1r_1 = Mr$$

the force P' is completely balanced, since the mass M_1 is exerting a centrifugal force equal to $M_1r_1\omega_1^2$ in the direction OM_1, and this has a horizontal component of $M_1r_1\omega_1^2 \cos \theta$.

The vertical component $M_1r_1\omega_1^2 \sin \theta$ of the centrifugal force of M_1 is not balanced, however. This force is a maximum at the top and bottom points, where it is equal to $M_1r_1\omega_1^2$. If the value of M_1r_1 were half that of Mr, one-half of the horizontal force P' would be balanced and the vertical force at the top and bottom positions would be only $\frac{1}{2}Mr\omega_1^2$. Since generally the horizontal force is more injurious to the mechanism than the vertical, the usual practice is to balance about two-thirds of the horizontal force. This leaves an unbalanced horizontal force of $\frac{1}{3}Mr\omega_1^2$ and gives an unbalanced vertical force of $\frac{2}{3}Mr\omega_1^2$.

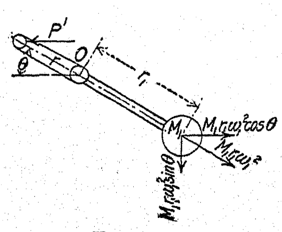


FIG. 420.

In the ordinary reciprocating engine with connecting rod, the acceleration of the piston at the head end of the cylinder is greater than $r\omega_1^2$ by the amount $l\omega^2 = l\frac{v^2}{l^2} = \frac{r}{l}(r\omega_1^2)$ and at the crank end is less than $r\omega_1^2$ by the same amount, as may be shown by the method of Art. 150. The mean value is $r\omega_1^2$, so the reciprocating parts of such an engine are balanced in the same manner as those of the slotted slider.

Problems

1. A single-cylinder horizontal engine has reciprocating parts weighing 800 lb., a crank 10 in. long, and two crank webs. If it is balanced according to usual practice, how much weight will be required on each crank web at a distance of 8 in. from the axis? *Ans.* 333 lb.

2. If the engine referred to in Prob. 1 is running at 240 r.p.m., what is the maximum unbalanced vertical force? *Ans.* 8730 lb.

154. Balancing Both Rotating and Reciprocating Parts.—In the usual type of engine with connecting rod, the rod has a combined rotation and translation. For the purposes of balancing,

the small crosshead end of the rod and one-half the plain part of the rod are considered to have a motion of translation with the crosshead, crosshead pin, piston rod, and piston. The large crank end of the rod and the remaining half of the plain part of the rod are considered to have a motion of rotation with the crank and crankpin. In ordinary steam-engine construction the former is about one-third and the latter two-thirds of the weight of the rod.

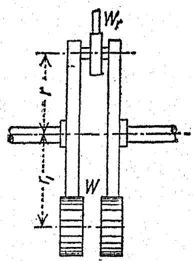


FIG. 421.

If the moving parts are to be balanced in the plane of the crank, the following relation applies. Let W_t be the weight of the reciprocating parts, consisting of the piston, piston rod, crosshead, and one-third of the connecting rod. Let W_r be the weight of the rotating parts, consisting of the crank, crankpin, and two-thirds of the connecting rod. Let W be the weight of the counterbalance at radius r_1 and let r be the length of the crank. Then from Arts. 142 and 153,

$$Wr_1 = W_r r + \tfrac{2}{3} W_t r$$

If there are two crank webs as shown in Fig. 421, half of W must be in the plane of each.

Problems

1. The piston of a steam engine weighs 320 lb.; the piston rod weighs 90 lb.; the crosshead weighs 60 lb.; the connecting rod weighs 300 lb.; and the crankpin weighs 25 lb. Let $r_1 = r = 1$ ft., and let the arm of the counterweight balance the crank arm in each case. If the construction is such as that shown in Fig. 421, what is the counterweight necessary in each crank web? *Ans.* 302.5 lb.

2. At what speed will the engine described in Prob. 1 have an unbalanced vertical force of 4000 lb.? *Ans.* 173.5 r.p.m.

155. Balancing of Locomotives.—Figure 422 shows one of the simplest cases in the balancing of locomotives, that of an outside cylinder locomotive with four driving wheels. The crank B is a part of the forward, or main, driving wheel, and by means of the side rod BA the pressure from the connecting rod CB is transmitted to the crankpin of the rear wheel.

The motion is one of combined translation and rotation, and the effects of the reciprocating and rotating parts upon the frame of the locomotive are the same as if it were running upon a

stationary testing table. The conditions for balancing are the same as those for the stationary engine discussed in Art. 154, except that the parts to be balanced are in two separate planes and the counterweights must be added in still other planes. In this case it is necessary[1] to compute the balancing weights in the planes of the wheels on both sides of the locomotive, as

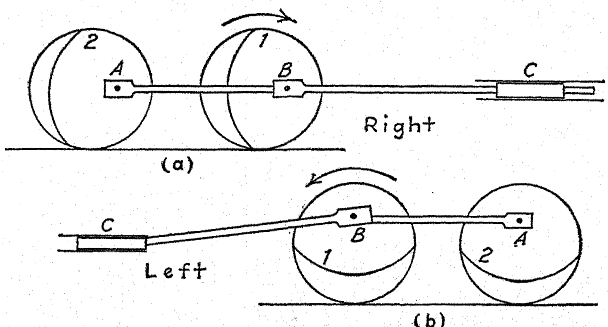


FIG. 422.

discussed in Art. 142. The two computed counterweights in each wheel may then be replaced by a single weight between the two.

For wheel 1 on each side, the rotating parts to be balanced consist of the large crankpin, one-half of the side rod, and two-thirds of the connecting rod. For wheel 2 on each side, the rotating parts to be balanced consist of the smaller crankpin and one-half of the side rod. The reciprocating parts consist of the piston, piston rod, crosshead, and one-third of the connecting rod. As in stationary engines, it is customary to counterbalance only a part of the reciprocating weight by rotating parts, as explained in Art. 153. In practice this amount varies from one-half to two-thirds of the weight of the reciprocating parts. There are two methods of providing for the balancing of the reciprocating parts. One is to place all the counterweight for the reciprocating parts upon wheel 1, combined with the counterweight for its rotating parts. The other is to divide the counter-

[1] Before distances between cylinders became so great, locomotive drivers were counterbalanced as though the rotating parts to be balanced were in the same plane as the counterbalancing weights, that is, the wheels. Drivers so balanced are in *static* balance but not in kinetic balance. If allowance is made for the fact that the plane of the rotating parts is not the same as that of the counterweight, such balancing is called *cross* balancing.

weight for the reciprocating parts equally between the two wheels. With the first method, the large unbalanced vertical kinetic force due to the heavy counterweight on wheel 1 produces excessive pressures upon the rail, so, in general, the second method is to be preferred.

EXAMPLE

For a locomotive of the type shown in Fig. 422, compute the amounts and positions of the counterweights necessary to balance each wheel, given the following dimensions: crank, 1 ft.; diameter of wheels, 6 ft.; weight of crankpin on wheel 1, 60 lb., with center of gravity 7.75 in. from the plane of the wheels; weight of crankpin on wheel 2, 40 lb., with center of gravity 5.75 in. from the plane of the wheels; weight of side rod, 240 lb., also 5.75 in. from the plane of the wheels; weight of connecting rod, 280 lb., with center of gravity 10 in. from the plane of the wheels; weight of crosshead, 154 lb.; weight of piston and piston rod, 306 lb.; distance between central planes of wheels, 59 in. Counterbalance two-thirds of the reciprocating parts.

Solution.—Consider the wheels on the right side, the piston being at the forward end of its stroke. The reciprocating weights are

$$306 + 154 + \tfrac{1}{3} \times 280 = 553.33 \text{ lb.}$$

The part of this which is to be balanced by rotating counterweights is

$$\tfrac{2}{3} \times 553.33 = 368.88 \text{ lb.}$$

The counterweight for this amount of the reciprocating parts will be divided equally between the front and rear wheels, 184.44 lb. being balanced in each.

For wheel 1, the rotating parts are three in number: two-thirds of the connecting rod, the crankpin, and one-half of the side rod. The total weight to be balanced on wheel 1 and their distances from the plane of the wheel are as follows:

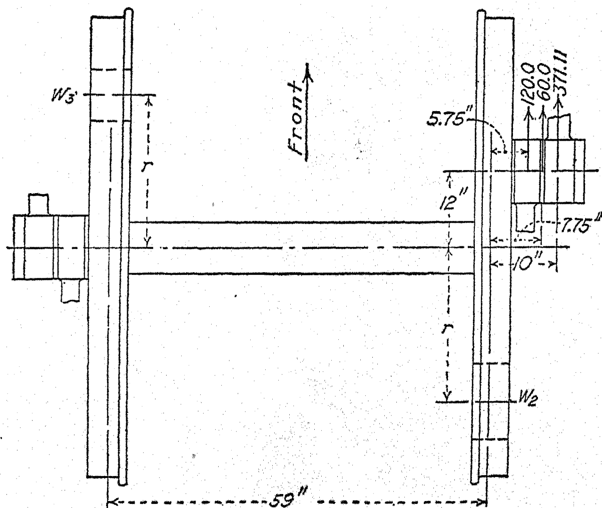
At 10 in., $184.44 + \tfrac{2}{3} \times 280 = 371.11$ lb.
At 7.75 in., 60 lb.
At 5.75 in., 120 lb.

Let W_2 be the counterweight in the wheel on the right side, W_3 the counterweight in the wheel on the left side, and r the radius of each counterweight, as shown in Fig. 423. Then, as in Fig. 392(*b*), since the radius of the crank is 1 ft.,

$$\begin{aligned}(W_2r \times 59) &= (120 \times 64.75) + (60 \times 66.75) + (371.11 \times 69)\\ W_2r &= 633.6 \text{ lb.-ft.}\\ (W_3r \times 59) &= (120 \times 5.75) + (60 \times 7.75) + (371.11 \times 10)\\ W_3r &= 82.5 \text{ lb.-ft.}\end{aligned}$$

The counterweight W_2 on the wheel on the right side is at the rear; the counterweight W_3 on the wheel on the left side is at the front. The crank on

the wheel on the left side is 90° behind that on the right side, so that when the crank on the right side is in the forward position as shown in Fig. 423, that on the left side is above the axle at the highest point in its travel. Let



Top View
FIG. 423.

W_2' be the counterweight on the left wheel to balance the weights on the left crankpin, and W_3' the counterweight on the right wheel to balance the weights on the left crankpin. Then W_3' will be directly above the axle, as shown in Fig. 424, and $W_3'r = 82.5$ lb.-ft. As explained in Art. 142, W_2 and W_3' may be replaced by a single weight W, such that $Wr = \sqrt{633.6^2 + 82.5^2}$.

$$Wr = 639 \text{ lb.-ft.}$$

If $r = 2$ ft.,

$$W = 319.5 \text{ lb.}$$

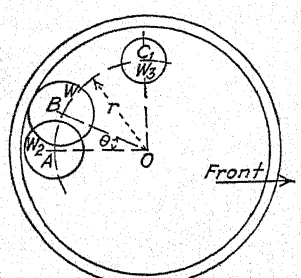
The angle θ that the radius OB makes with the horizontal radius OA is

$$\theta = \tan^{-1} \frac{82.5}{633.6}$$

$$\theta = 7°25'$$

$$\text{Arc } AB = 2 \times \frac{7.4}{57.3} = 0.258 \text{ ft.}$$

$$AB = 3.10 \text{ in.}$$



Right Wheel from Right Side
FIG. 424.

In the front wheel on the left side, the counterweight will be the same in amount, located below the axle, 7°25′ forward from the vertical radius. It is seen that, whereas the crank on the left is 90° behind the crank on the right, the counterweight on the left is 104°50′ behind that on the right.

If h is negative (upward), the velocity of the body is decreased. The normal constraining force is given by the equation

$$N = \pm W \cos \theta + \frac{W}{g}\frac{v^2}{r},$$

r being the radius of curvature of the path. The negative sign is used if the body is above a horizontal line through the center of curvature.

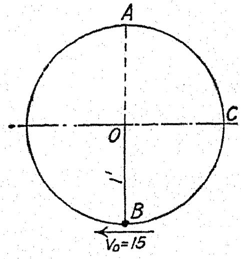


FIG. 427.

EXAMPLE

A body B weighing 2 lb. is rotating in a vertical circle at the end of a cord 1 ft. long, as shown in Fig. 427. If its velocity at the bottom point is 15 ft./sec., what is its velocity at the top point A? What is the tension T in the cord at that point? What is the least velocity it can have at point A which will keep it in its circular path?

Solution.

$$v_A^2 = v_0^2 - 2gh = 225 - 128.8$$
$$v_A^2 = 96.2$$
$$v_A = 9.81 \text{ ft./sec.}$$

When the body is at point A, the only forces acting upon it are its weight and the tension in the cord, both downward. These produce the normal acceleration $a_n = v^2/r = 96.2$ ft./sec.².

From the equation $F = \frac{W}{g}a$,

$$T + 2 = \frac{2}{32.2} \times 96.2$$
$$T = 3.98 \text{ lb.}$$

Since the cord cannot have a compressive stress, the minimum value that T can have is zero. When T is zero, equation $F = \frac{W}{g}\frac{v^2}{r}$ gives

$$2 = \frac{2}{32.2} \times \frac{v^2}{1}$$
$$v = 5.67 \text{ ft./sec.}$$

This is the least velocity that will keep the body in its circular path.

Problems

1. If the body shown in Fig. 427 starts from rest at point C, what is its velocity as it passes the 30° point? What is its velocity at the bottom point? *Ans.* 5.68 ft./sec.; 8.03 ft./sec.

2. If a body weighing 5 lb. is rotating in a vertical circle at the end of a cord 6 ft. long, what is the least velocity at the bottom that will keep it in

the circular path at the top? What is the tension in the cord when the body is at the bottom? When it is 60° from the top? When it is 45° from the bottom? *Ans.* 31.08 ft./sec.; 30 lb.; 7.5 lb.; 25.6 lb.

3. From an airplane traveling horizontally at a speed of 150 m.p.h. at an elevation of 800 ft., a ball is thrown laterally at a speed of 80 ft./sec. with respect to the airplane. Neglecting air resistance, compute its speed as it strikes the ground. *Ans.* 326 ft./sec.

4. If friction and air resistance are neglected, compute the minimum height h, Fig. 428, from which a car may start in order to remain in contact with the track at A. The center of gravity of the car and its load is a distance e from the plane through the base of the wheels. *Ans.* $h = 2.5r - 1.5e$.

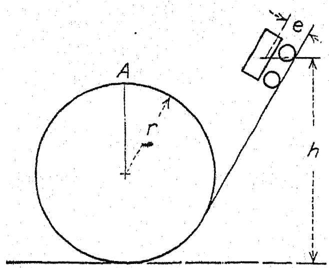


Fig. 428.

5. If friction and air resistance are neglected, get the height h, Fig. 428, from which a car must start in order that the pressure of the track on the car at A shall just equal its weight. *Ans.* $h = 3r - 2e$.

6. If in Fig. 428 the height $h = 24$ ft., $r = 7.5$ ft., $e = 1.5$ ft., and the weight of the car is 400 lb., get the normal pressure of the track on the car as it passes point A. Get the normal pressure of the track on the car as it passes the point 45° from point A. *Ans.* 1000 lb.; 1352 lb.

157. Motion of Projectile.—A projectile is a body that is given an initial velocity and then moves with only the resistance of the air and the action of gravity upon it. Since the resistance

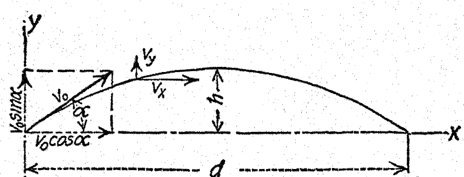


Fig. 429.

of the air is different for each different shape and size of projectile, it cannot be taken into consideration in a general solution and is therefore neglected. The equations of the general solution must be modified properly for each different size and shape of projectile. As derived, they are for the motion of a particle in a vacuum.

If the initial velocity of the projectile v_0, Fig. 429, is resolved into its horizontal and vertical components $v_0 \cos \alpha$ and $v_0 \sin \alpha$, respectively, the two component motions may be analyzed

separately. Since no horizontal force is acting upon the body, there is no change in horizontal velocity. The horizontal distance moved in time t is

$$x = v_0 t \cos \alpha$$

In the vertical direction, the only force acting is its own weight, so its motion is the same as that of a falling body.

$$v_y = v_0 \sin \alpha - gt$$
$$y = v_0 t \sin \alpha - \tfrac{1}{2}gt^2$$

If the equation of the path is desired, it is obtained by eliminating t between the equations for x and y.

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}$$

This is the equation of a parabola with its axis vertical.

In the solution of problems, it is usually simpler to consider the component motions separately and to work by analysis rather than by substitution in the formulas.

EXAMPLE 1

A projectile is discharged with a velocity of 200 ft./sec. at an angle of 30° with the horizontal. Determine the maximum height, the time until it returns to the same level, and the range. Determine also the velocity at the end of 2 sec.

Solution.—The vertical component of the velocity is $v_0 \sin \alpha = 100$ ft./sec. The horizontal component of the velocity is $v_0 \cos \alpha = 173.2$ ft./sec. It loses vertical velocity at the rate of 32.2 ft./sec.², so its vertical velocity will be zero in time

$$t_1 = \frac{100}{32.2} = 3.106 \text{ sec.}$$

Its average vertical velocity is 50 ft./sec., so its height will be

$$h = 50 \times 3.106 = 155.3 \text{ ft.}$$

It requires as long a time to fall to the same level as it required for it to rise, so it is in the air a total time

$$t = 6.212 \text{ sec.}$$

The horizontal distance moved, the range, is

$$d = 173.2 \times 6.212 = 1075.9 \text{ ft.}$$

At the end of 2 sec. the vertical velocity has been reduced by 2×32.2 ft./sec. and is

$$v_y = 100 - 64.4 = 35.6 \text{ ft./sec.}$$

The horizontal velocity is unchanged. The resultant velocity is

$$v = \sqrt{35.6^2 + 173.2^2} = 176.8 \text{ ft./sec.}$$

The angle with the horizontal is

$$\theta = \tan^{-1} \frac{35.6}{173.2} = 11°37'$$

EXAMPLE 2

From a tower 200 ft. high a stone is thrown downward at an angle of 45° with the horizontal, with a velocity of 80 ft./sec. How far away from the tower does it strike the ground and what is the time required?

Solution.—The horizontal component of the velocity is $v_0 \cos \alpha = 56.56$ ft./sec. The initial vertical velocity downward is the same. The final vertical velocity after time t is $56.56 + 32.2t$. The average vertical velocity is $56.56 + 16.1t$ and the distance fallen is $56.56t + 16.1t^2$. Since this equals 200, $t = 2.18$ sec. In 2.18 sec. the horizontal distance moved is $56.56 \times 2.18 = 123.3$ ft.

Problems

1. A projectile is discharged with a velocity of 3000 ft./sec. at an angle of 5° above the horizontal. Get the height and the range.

Ans. $h = 1058$ ft.; $d = 48{,}450$ ft.

2. Prove that the value of the angle of elevation α to give a maximum range is 45°.

3. Compute the theoretical muzzle velocity required to give a projectile a range of 75 miles when the angle $\alpha = 50°$. Get also the maximum height.

Ans. 3600 ft./sec.; 22.3 miles.

4. With a muzzle velocity of 5500 ft./sec., what is the maximum theoretical range of a projectile? *Ans.* 178 miles.

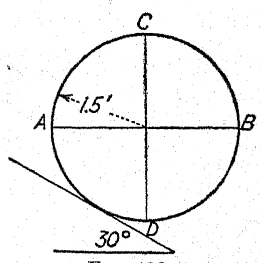


FIG. 430.

GENERAL PROBLEMS ON ANY PLANE MOTION OF RIGID BODIES

1. In Fig. 430, a solid circular cylinder 3 ft. in diameter is shown in the position that it has reached 2 sec. after being released from rest and allowed to roll freely down the plane. Get the absolute velocities of points A and B.

Ans. $v_A = 21.47$ ft./sec., 30° above H; $v_B = 37.2$ ft./sec., 60° below H.

2. Get the absolute accelerations of points C and D of the cylinder of Prob. 1.

Ans. $a_C = 314$ ft./sec.², 86°20′ below H; $a_D = 302$ ft./sec.², 89° 44′ with H.

3. With the directions of the velocities of points A and B known from the result of Prob. 1, locate the instantaneous center of the cylinder.

4. Using the instantaneous center as the point of reference, solve for the absolute acceleration of point A, Fig. 430.

Ans. 317 ft./sec.², 0°58′ with H.

5. The connecting rod shown in Fig. 431 is 5 ft. long and the crank is 1 ft. long. Begin at dead center with the connecting rod in the position AM and locate the position of the instantaneous center of the rod for each 30° of a half revolution.

Ans. AM, at A; BN, 3.37 ft. above B; CP, 9.4 ft. above C; DQ at infinity; ER, 7.66 ft. below E; FS, 2.37 ft. below F; GT at G.

6. If the crankpin of the connecting rod referred to in Prob. 5 is rotating clockwise at 200 r.p.m., get the velocity of the crosshead for each 30° point in the half revolution.

Ans. $v_A = 0$; $v_B = 12.3$ ft./sec.; $v_C = 20.0$ ft./sec.; $v_D = 20.94$ ft./sec.; $v_E = 16.3$ ft./sec.; $v_F = 8.64$ ft./sec.; $v_G = 0$.

7. The connecting rod shown in Fig. 431 weighs 220 lb., its center of gravity is 3.2 ft. from the crosshead pin, and its moment of inertia with respect to the axis of the crosshead pin is 96. The crankpin is rotating at 200 r.p.m. Get the crankpin and guide reactions for the two positions, as follows: (1) dead center at head end, horizontal pressure of crosshead pin 3000 lb.; (2) 60° from dead center, horizontal pressure of crosshead pin 8000 lb.

Ans. (1) $N = -216$ lb.; $T = 141$ lb. upward; $N_A = 79$ lb. (2) $N = 6960$ lb.; $T = 1460$ lb. downward; $N_A = 1200$ lb.

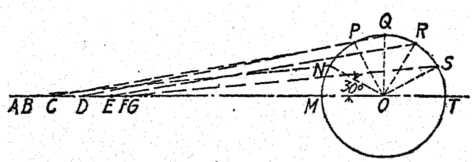


FIG. 431.

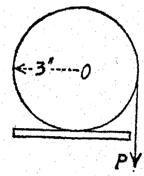


FIG. 432.

8. Solve for the reactions on the connecting rod of Prob. 7 when $\theta = 180°$. Force at crosshead pin is 7000 lb. acting to the left.

Ans. $N = 4226$ lb. to the right; $T = 141$ lb. upward; $N_A = 79$ lb.

9. A steel cylinder 6 in. in diameter and 5 ft. long rests with each end on a horizontal rail normal to the direction of its axis, as shown in Fig. 432. If static $f = 0.25$ and kinetic $f = 0.20$, determine the motion if a force $P = 140$ lb. is applied vertically downward to a rope wrapped around the cylinder at its middle.

Ans. Cylinder rolls to right; static $F = 93.4$ lb.; $a = 6.25$ ft./sec.².

10. Solve Prob. 9 if the rope is wrapped through 180° more and the force is applied vertically upward.

Ans. Cylinder slides to right and rotates clockwise; kinetic $F = 68.13$ lb.; $a = 4.56$ ft./sec.²; $\alpha = 38.4$ rad./sec.².

11. Solve Prob. 9 if the force pulls horizontally to the left on the rope at the bottom of the cylinder.

Ans. Cylinder slides to left and rotates clockwise; kinetic $F = 96.13$ lb.; $a = 2.94$ ft./sec.²; $\alpha = 23.5$ rad./sec.².

12. Solve Prob. 9 if the force pulls horizontally to the right at the top of the cylinder.

Ans. Cylinder rolls to the right; static $F = 46.8$ lb.; $a = 12.5$ ft./sec.²; $\alpha = 50$ rad./sec.².

13. Solve Prob. 9 if the force pulls downward and to the left at an angle of 45° with the horizontal.

Ans. Cylinder rolls to the right; static $F = 126.3$ lb.; $a = 1.83$ ft./sec.^{2}; $\alpha = 7.32$ rad./sec.^{2}.

14. If in Prob. 13 the angle of the force with the horizontal is decreased, for what value of the angle will slipping impend? What is the amount of the acceleration?

Ans. 25°50′; $a = 0.625$ ft./sec.^{2}.

15. Figure 433 represents a solid circular cylinder A weighing 100 lb., free to rotate about an axle through its geometric axis. A cord is fastened to the axle, passing up over the pulley B, the mass of which is neglected, then down and around the cylinder A, and finally down to the weight C of 120 lb. The friction of the cord around the cylinder is sufficient to prevent slipping. Solve for the linear and angular acceleration of cylinder A and for the tensions T_1 and T_2.

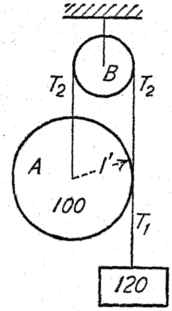


FIG. 433.

Ans. $a = 1.533$ ft./sec.^{2}; $\alpha = 3.067$ rad./sec.^{2}; $T_1 = 114.3$ lb.; $T_2 = 109.54$ lb.

16. A plane 15 ft. long is inclined at an angle of 15° with the horizontal. Static $f = 0.15$ and kinetic $f = 0.12$. If a cylinder 6 in. in diameter starts from rest at the top and rolls down, determine the time required for it to reach the bottom and its linear and angular velocity at the bottom.

Ans. $t = 2.32$ sec.; $v = 12.9$ ft./sec.; $\omega = 51.6$ rad./sec.

17. The plane described in Prob. 16 is raised to an angle of 24° with the horizontal. Two similar cylinders are released from rest at the top. The one slides on its base; the other rolls freely. Find the time required for each cylinder to reach the bottom.

Ans. One slides in 1.772 sec.; the other rolls in 1.855 sec.

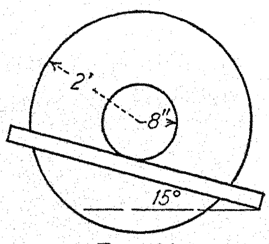


FIG. 434.

18. A sphere 12 in. in diameter, a solid circular cylinder 12 in. in diameter, and a hollow circular cylinder 12 in. outside diameter and 11 in. inside diameter all roll freely down a 30° plane 25 ft. long. Get the time of each and the linear velocity of each at the bottom.

Ans. Sphere, $t = 2.09$ sec.; $v = 24$ ft./sec. Cylinder, $t = 2.16$ sec.; $v = 23.2$ ft./sec. Hollow cylinder, $t = 2.44$ sec.; $v = 20.5$ ft./sec.

19. In Fig. 434, the larger cylinder is steel, 4 ft. in diameter and 2 in. thick. The two small cylinders upon which the assembly rolls on the rails are each 16 in. in diameter and 3 in. thick. Get the frictional force at the rails, assuming free rolling. Get the time for it to move 10 ft. from rest.

Ans. $F = 275$ lb.; $t = 3.28$ sec.

20. Assume that in the engine described in Prob. 1, Art. 154, the cranks are replaced by disks that need no balancing. Instead of the balance weights shown in Fig. 421, balancing is to be effected by adding one counterweight to the rim of a large flywheel 30 in. to the left of the plane of the

connecting rod and another to the rim of a small flywheel 24 in. to the right of the plane of the connecting rod. If the radius of the counterweight in the large flywheel is 5.5 ft. and in the small flywheel is 2.5 ft., find the weights necessary. *Ans.* 48.9 lb.; 134.4 lb.

21. A mountain-type locomotive has eight drive wheels, the connecting rod being fastened to wheel 2. The drivers are 70 in. in diameter; the cylinders are 93 in. center to center and 30 in. long; the side rods are 77 in. center to center; and the planes of the drivers are 60 in. center to center. The weight of the reciprocating parts on one side is 2550 lb. On the main driver the weight of the rotating parts in the plane of the connecting rod is 1040 lb.; that of the rotating parts in the plane of the side rod is 960 lb. On wheels 1 and 4, the weight of the rotating parts in the plane of the side rod is 260 lb. on each. On wheel 3, the weight of the rotating parts in the plane of the side rod is 660 lb. Counterbalance one-half of the weight of the reciprocating parts, divided equally among the four wheels. Compute the counterbalance for each wheel, assuming 24 in. as the radial distance of the counterweights in wheels 1 and 4, 20 in. in wheel 3, and 18 in. in wheel 2. The crank on the right side leads that on the left by 90°.

Ans. Wheels 1 and 4, 446 lb., at 10°0′ with crank diameter; wheel 2, 2396 lb., 10°10′ with crank diameter; wheel 3, 880 lb., 8°50′ with crank diameter.

22. When the locomotive described in Prob. 21 is running at 60 m.p.h., what is the kinetic effect of each driving wheel upon the track due to the counterbalance for the reciprocating parts? *Ans.* 14,680 lb.

23. A small block slides freely down the quadrant shown in Fig. 435. Determine the distance x_1, the equation of the path BC, and the velocity at C. *Ans.* $x = 9.8$ ft.; $y = -x^2/12$; 26.6 ft./sec.

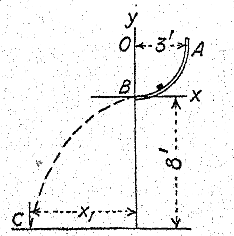


FIG. 435.

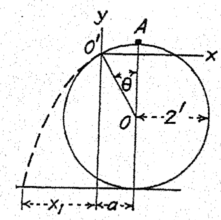


FIG. 436.

24. A block starts from rest at the top of a cylinder 4 ft. in diameter (Fig. 436) and slides without friction to point O' where it leaves the surface of the cylinder. Determine angle θ. Compute distances a and x_1.

Ans. 48°11′; 1.49 ft.; 1.43 ft.

25. If a car weighing 400 lb. starts from rest at the top of the incline (Fig. 437), what is the normal pressure of the track on the car at the top of the loop? If the track from the bottom of the loop rises 2 ft. in a distance of 8 ft., how wide a gap mn can be leaped? *Ans.* 2800 lb.; 26.4 ft.

26. If a target is distant 1800 ft. horizontally and 600 ft. higher than the gun, what angle of elevation is necessary if the muzzle velocity is 2000 ft./sec.? *Ans.* 18°50′.

27. The muzzle velocity of a projectile is 1600 ft./sec., and the distance of the target is 2 miles. What is the required angle of elevation of the gun?

Ans. 3°49′.

28. Figure 438 represents diagrammatically a ball-bearing exhibit. At intervals, small ball bearings are projected horizontally from the opening

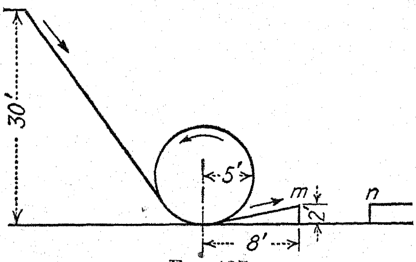


Fig. 437.

at A with a velocity of 2.59 ft./sec. The ball strikes a heavy polished base at B, 1.5 ft. below the point of discharge, from which it rebounds. The base is set at a small angle with the horizontal, so that the vertical plane through the trajectory of the rebound is at a small angle with the vertical plane

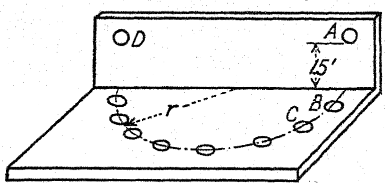


Fig. 438.

through the trajectory with which it struck. It then strikes the next base at C and rebounds in like manner, and so on for a total of eight points around the semicircle, finally disappearing into the reservoir at D. Neglecting the losses due to air resistance and impact, compute the chord distance between the points of impact around the circle, the radius of the circle, and the angle of inclination of each base with the horizontal.

Ans. 1.58 ft.; 4.06 ft.; 2°57′.

CHAPTER XVI

WORK, ENERGY, AND POWER

158. Work and Energy Defined.—The *work* done by a force is given by the product of the force and the distance through which the point of application of the force moves in the direction of the force. In Fig. 439(a), four forces F_1, W, N, and F are shown acting upon a body that is moving along a horizontal plane

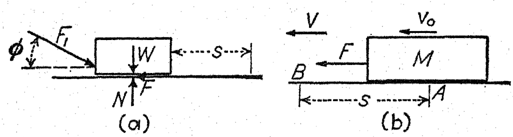


FIG. 439.

surface. The forces W and N do no work upon the body, since their points of application do not move in the direction of the forces. While the body moves the distance s, the point of application of force F_1 moves a distance $s \cos \phi$ in the direction of F_1, so the work done by force F_1 is $F_1 \times s \cos \phi$.

The quantity $F_1 \times s \cos \phi$ may also be written $F_1 \cos \phi \times s$. The quantity $F_1 \cos \phi$ is the component of F_1 in the direction of motion of its point of application, so it may also be stated that the work done by a force is equal to the product of the component of the force in the direction of motion of its point of application and the distance through which the point of application moves. The component $F_1 \cos \phi$ is called the *working component* of the force. The other component, $F_1 \sin \phi$, does no work.

If the displacement is in the same direction as the working component of the force, the work is positive, as in the case of F_1 above, and the work is said to be done *by the force*. If the displacement is in the direction opposite to the working component of the force, the work is negative, as in the case of the frictional force F above. The work is said to be done *against the force* and is equal to $-Fs$.

If the force is variable, the work done in a small distance ds is $F \cos \phi \; ds$ as before, ϕ being the angle between the direction of the motion. Let U represent the work.

Then the total amount of the work done by the force F is given by the expression

$$U = \int F \cos \phi \, ds$$

If the relation between F, ϕ, and s is known, this expression can be integrated.

The work done on a particle against gravity in raising it through any distance is equal to the weight w of the particle and the vertical distance h through which it is raised, irrespective of its lateral motion.

$$U = wh$$

If a particle is lowered, gravity does positive work upon it, and the amount of the work is equal to the product of the weight of the particle and the vertical displacement as before.

If a body of weight W is composed of differential parts, each with weight dW, which are raised different distances y against gravity, the total work done on the body is given by the expression

$$U = \int y \, dW = \bar{y}W$$

The quantity $\bar{y}$ is the vertical distance h through which the center of gravity of the body is raised, so

$$U = Wh$$

The *unit of work* is the work of one unit of force through one unit of distance. In the English system, the *foot-pound* is the unit most generally used.

Energy is the capacity to do work. If a weight has capacity to do work on account of its position above a chosen datum plane, its energy is called *potential energy*. If released from its support, it may be made to do positive work in descending to its zero position. The steam in a boiler, a charged storage battery, and a loaded spring are said to have potential energy due to their stressed condition.

A moving body, by virtue of its velocity, has capacity to do work as it is brought to rest. This energy of motion is called *kinetic energy*.

Problems

1. How much work is done against gravity in pulling a 120,000-lb. car at a uniform speed of 8 m.p.h. up a 1.5 per cent grade a distance of 1 mile?

If train resistance is 8 lb. per ton, what is the total work done by the drawbar pull? *Ans.* 9,504,000 ft.-lb.; 12,038,400 ft.-lb.

2. A chain 80 ft. long weighing 6 lb./lin. ft. passes over a pulley with 20 ft. on one side and 60 ft. on the other. What work is done against gravity as the pulley is rotated until the middle of the chain is at the pulley? From this position, what work is done by gravity as the pulley is rotated until the end of the chain reaches the pulley? *Ans.* 2400 ft.-lb.; 9600 ft.-lb.

3. A vertical mine shaft 8 ft. square is driven through 60 ft. of clay, 50 ft. of shale, 80 ft. of sandstone, and 200 ft. of granite. The material is raised 10 ft. above the mouth of the shaft. If clay weighs 110 lb./cu. ft., shale 120 lb./cu. ft., sandstone 150 lb./cu. ft., and granite 165 lb./cu. ft., what is the total amount of work done? *Ans.* 404,928 ft.-tons.

159. Relation between Work and Kinetic Energy.—Since kinetic energy is the capacity of a body to do work on account of its motion or velocity, the *amount* of its kinetic energy is necessarily equal to the amount of work done by the positive acting force or forces in producing that velocity. Let F be the resultant force that acts upon a particle of mass m to produce the velocity v in the distance s. Then the work done by the resultant force is equal to

$$U = \int_0^s F\,ds$$

Since $F = Ma$,

$$U = \int_0^s ma\,ds$$

Since $a\,ds = v\,dv$,

$$U = \int_0^v mv\,dv$$

$$U = \tfrac{1}{2}mv^2$$

In terms of the velocity of the particle, the work done by the resultant force F is $\frac{1}{2}mv^2$. The quantity $\frac{1}{2}mv^2$ is called the *kinetic energy* of the particle. Kinetic energy is commonly abbreviated K.E., so

$$\text{K.E.} = \tfrac{1}{2}mv^2$$

Problems

1. A bullet weighing 0.2 oz. is shot from a rifle with a muzzle velocity of 4200 ft./sec. What is its kinetic energy? *Ans.* 3424 ft.-lb.

2. With what velocity must a 10-lb. rifle be moving to have the same kinetic energy as the bullet in Prob. 1? *Ans.* 148.5 ft./sec.

160. Kinetic Energy of Translation: Forces Constant.—The kinetic energy of a body at any instant is equal to the sum of the kinetic energies of the particles of which it is composed. In a motion of translation, each particle of the body has the same velocity at any instant, so the total kinetic energy is

$$\text{K.E.} = \Sigma \tfrac{1}{2} m v^2 = \tfrac{1}{2} v^2 \Sigma m = \tfrac{1}{2} M v^2$$

M being the mass of the whole body.

Let F, Fig. 439(b), be the resultant of all the working forces acting upon a body of mass M as it moves from A to B through the distance s. Let v_0 be its velocity at A and v its velocity at B. Then

$$U = \int_0^s F\, ds = \int_{v_0}^{v} Mv\, dv$$

$$\mathbf{U} = \int_0^s \mathbf{F} \mathbf{ds} = \tfrac{1}{2}\mathbf{M}\mathbf{v}^2 - \tfrac{1}{2}\mathbf{M}\mathbf{v}_0^2$$

$\tfrac{1}{2}Mv_0^2$ is the kinetic energy of the body at A, and $\tfrac{1}{2}Mv^2$ is its kinetic energy at B, so the following general statement may be made:

In any motion of translation, the positive work done by the resultant force is equal to the increase in kinetic energy.

If F is constant, $\int_0^s F\, ds = Fs$, so

$$\mathbf{U} = \mathbf{Fs} = \tfrac{1}{2}\mathbf{M}\mathbf{v}^2 - \tfrac{1}{2}\mathbf{M}\mathbf{v}_0^2$$

It is usually simpler in any given problem to consider separately the work of the several forces acting upon the body instead of the work of their resultant. Let F_1, F_2 . . . be the forces, and F_1', F_2' . . . their components in the direction of the motion of the body.

$$U = \int F_1'\, ds + \int F_2'\, ds + \cdots$$

If the forces are constant and act through distances s_1, s_2, etc., respectively, in the direction of the motion of the body,

$$U = F_1' s_1 + F_2' s_2 + \cdots$$

If some of the forces are resistances, their work is negative.

In any motion of translation the work done by the positive forces minus the work done by the negative forces is equal to the increase in kinetic energy.

This may be written

Positive work − negative work = final K.E. − initial K.E.

If the term $\frac{1}{2}Mv_0^2$ is transferred to the other side of the equation, it may be written

Initial K.E. + positive work − negative work = final K.E.

EXAMPLE

An 80,000-lb. car is hauled up a 2 per cent incline by a constant draw bar pull of 1000 lb. If the train resistance is 6 lb./ton and the initial velocity is 20 ft./sec., how far up will it go before its velocity is reduced to 10 ft./sec.?

Solution.—Initial K.E. $= \frac{1}{2}Mv_0^2 = \frac{1}{2}\times\frac{80,000}{32.2} \times 20^2 = 497,000$ ft.-lb.

Positive work $= 1000s$, if s is the distance in feet.
Negative work of train resistance $= 240s$.

Negative work of gravity $= \frac{80,000}{50}s = 1600s$

Final K.E. $= \frac{1}{2}Mv^2 = \frac{1}{2}\times\frac{80,000}{32.2} \times 10^2 = 124,200$ ft.-lb.

Then

$$497,000 + 1000s - 240s - 1600s = 124,200$$
$$840s = 372,800$$
$$s = 444 \text{ ft.}$$

Problems

1. If when the velocity of the car in the foregoing example is 10 ft./sec. the drawbar pull is increased to 1500 lb., how much farther will the car go before coming to rest? *Ans.* 365 ft.

2. If when the car of Prob. 1 has come to rest the drawbar pull is removed and the car is allowed to run back down the grade, what will be its velocity at the lower end of the grade? If the track is then level, how far out on the level track will the car run before coming to rest?

Ans. $v = 29.7$ ft./sec.; 4580 ft.

3. A block of ice weighing 300 lb. rests on a level floor for which the coefficient of friction $f = 0.1$. What force applied downward and forward at an angle of 30° with the horizontal will give the block a velocity of 4 ft./sec. in a distance of 6 ft.? *Ans.* 52 lb.

4. A coal train consisting of 50 cars, each weighing 140,000 lb., is being pulled up a one-half of 1 per cent grade 15,000 ft. long by a constant drawbar pull of 60,000 lb. If its initial velocity is 30 m.p.h., and train resistance is 8 lb./ton, what will be its velocity at the top of the grade?

Ans. 26.6 m.p.h.

161. Kinetic Energy of Translation: Forces Variable.—If some of the forces acting upon a body during its motion are vari-

able, the relation between work and change in kinetic energy becomes

$$U = \int F\,ds = \tfrac{1}{2}Mv^2 - \tfrac{1}{2}Mv_0^2$$

If F varies with s, it must first be expressed in terms of s if the law of its variation is known, and then the expression integrated.

If a spring is deformed, the resistance that it offers is proportional to the amount of its deformation. Hence, if a spring is the means of applying a force to a body, the amount of the force will be a constant C multiplied by the distance deformed s, or $F = Cs$. The work done by the force is

$$\int_0^s Cs\,ds = \frac{Cs^2}{2} = \frac{F}{2}s$$

This is the product of the average force and the distance.

If one of the forces is steam, working expansively, theoretically the absolute pressure varies inversely as the distance from the end of the cylinder, so the amount of the force during expansion is equal to a constant C_1 divided by the distance s from the end of the cylinder, or $F = C_1/s$. An example of each of the cases mentioned will better illustrate the method of solution.

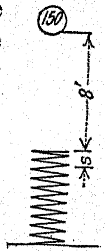


FIG. 440.

EXAMPLE 1

A body weighing 150 lb. falls 8 ft. from rest and strikes a 2000-lb. spring. What is the deformation of the spring?

Solution.—The body and spring are shown in Fig. 440. The body is at rest before starting to fall, hence has zero kinetic energy. When the spring has its maximum compression, the body is again at rest and has zero kinetic energy. The resistance of the spring is 2000 × displacement in inches, or 24,000 × displacement in feet. If s is the displacement in feet, the resistance of the spring is 24,000s. Then

$$150 \times 8 + 150s - \int_0^s 24{,}000s\,ds = 0$$
$$1200 + 150s = 12{,}000s^2$$
$$s = 0.3224 \text{ ft.} = 3.87 \text{ in.}$$

The velocity at any point, as when the compression is 1 in., may be found by equating the resultant work to the kinetic energy.

$$150 \times 8.083 - \int_0^{1/12} 24{,}000s\,ds = \frac{1}{2}\frac{150}{32.2}v^2$$
$$v = 22 \text{ ft./sec.}$$

EXAMPLE 2

Figure 441(a) represents a steam hammer. The ram and piston weigh 500 lb. The piston is forced downward by a steam pressure of 10,000 lb. against an air resistance of 1000 lb. The stroke s_2 is 18 in. Considering cutoff when $s_1 = 6$ in., find the velocity of the hammer as it strikes the metal, assuming pressure × volume = constant for steam after cutoff, and neglecting clearance volume and friction.

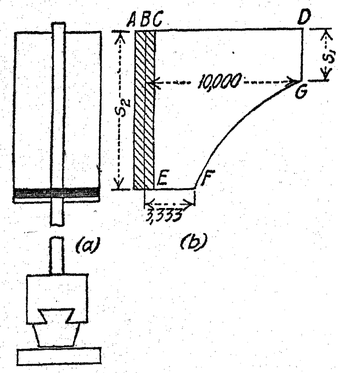


FIG. 441.

Solution.—The work of the positive forces minus the work of the resisting forces is equal to the increase in kinetic energy. The weight and the steam pressure are the positive forces; the atmospheric pressure is the resisting force; and its total kinetic energy at striking is the increase in kinetic energy, since $v_0 = 0$.

In Fig. 441(b), ordinate DB represents the initial steam pressure P_1; ordinate BA represents the force of gravity; and ordinate AC represents the resisting force of the air. Then area $DCEFG$ represents the resultant work done on the hammer till the point of striking. (See Art. 165.) The work of gravity $= 500 \times 1\frac{1}{2} = 750$ ft.-lb. The work of steam to cutoff $= P_1 s_1 = 10{,}000 \times \frac{1}{2} = 5000$ ft.-lb. The work of steam after cutoff $= \int_{s_1}^{s_2} P\,ds$, P being the variable pressure. Since the pressure varies inversely as the volume, it also varies inversely as the distance from the end of the cylinder, or

$$\frac{P}{P_1} = \frac{s_1}{s}$$

$$P = \frac{P_1 s_1}{s} = \frac{5000}{s}$$

Then work of steam after cutoff $= 5000 \int_{0.5}^{1.5} \frac{ds}{s}$

$$= 5000 \log_e \frac{1.5}{0.5}$$
$$= 5000 \log_e 3$$
$$= 5000 \times 2.3 \log_{10} 3$$
$$= 5000 \times 2.3 \times 0.477$$
$$= 5490 \text{ ft.-lb.}$$

The negative work of atmospheric pressure $= 1000 \times 1.5 = 1500$ ft.-lb

Work of gravity + work of steam − work of air $= \frac{1}{2}Mv^2$

$$750 + 5000 + 5490 - 1500 = \frac{1}{2} \times \frac{500}{32.2}v^2$$

$$v = 35.4 \text{ ft./sec.}$$

Problems

1. A body weighing 80 lb. starts from rest at the top of a plane at an angle of 60° with the horizontal and, after sliding 10 ft. down the plane, strikes a 150-lb. spring. If the coefficient of friction $f = 0.3$, and the body is in contact with the plane through its entire motion, how much is the spring compressed? *Ans.* 9.96 in.

2. A body weighing 10 lb. is projected vertically upward with an initial velocity of 40 ft./sec. At a point 8 ft. above the point of discharge, it strikes a coil spring that has a scale of 300 lb./in. Get the velocity with which it strikes the spring and the amount the spring is compressed.
Ans. 32.9 ft./sec.; 3.6 in.

3. A freight car weighing 100,000 lb. is moving with a velocity of 3 ft./sec. when it strikes a bumping post. Assuming the drawbar spring to take all the compression, what must be the scale of the spring in order that the compression of the spring shall not exceed 4 in.? *Ans.* 20,960 lb./in.

4. A steam hammer has the following dimensions: weight of piston and ram, 2000 lb.; diameter of piston, 15 in.; diameter of piston rod, 2 in.; stroke, 36 in.; absolute steam pressure, 100 lb./in.^{2}; air pressure below piston, 15 lb./in.^{2}; cutoff at quarter stroke; expansion isothermal. Neglecting clearance volume and friction, determine the velocity of striking.
Ans. 30.6 ft./sec.

162. Kinetic Energy of Rotation.—Let Fig. 442 represent any body rotating about axis O with angular velocity ω. The velocity of any particle of mass dM at a distance ρ from the axis is $\rho\omega$, and its kinetic energy is $\frac{1}{2}dM\rho^2\omega^2$. The total kinetic energy of the body is then $\int \frac{1}{2}dM\rho^2\omega^2$. Since ω^2 for all the particles at any instant is the same, and since $\int dM\rho^2 = I_0$,

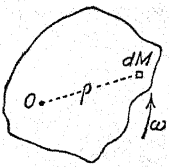


FIG. 442.

$$\text{K.E. of rotation} = \tfrac{1}{2}I_0\omega^2$$

The value of I_0 may be computed by the regular methods of integration if the form of the rotating body is regular. If not, it may be determined by experiment, as in Art. 116.

In order to determine the relation between work and kinetic energy of rotation, the same principle is made use of as in the case of kinetic energy of translation (Art. 160). If the force F is tangential, as in Fig. 443(a), the work done as force F moves through distance ds is $Fds = Fr\,d\theta$. If the force F is not tangential, the tangential component alone does work. Let F, Fig. 443(b), be the resultant of all the forces acting upon the body

except the reaction at O; and let ϕ be the angle that it makes with the tangent to the circle of rotation. When the point of application of force F is moved a distance $ds = r\,d\theta$, the work done by force F is

$$dU = F \cos \phi \, r \, d\theta$$

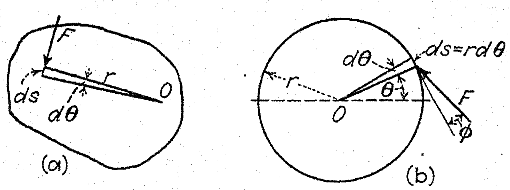


FIG. 443.

The quantity $Fr \cos \phi$ is the torque about the axis of rotation O. Let this torque be represented by C. The total work U is given by the equation

$$U = \int C \, d\theta$$

Since $C = I\alpha$ and $\alpha \, d\theta = \omega \, d\omega$,

$$U = \int C \, d\theta = \int I\alpha \, d\theta = \int_{\omega_0}^{\omega} I\omega \, d\omega$$

By integration,

$$U = \int C \, d\theta = \tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_0^2$$

The quantity $\int C \, d\theta$ is the work done by the resultant rotating force; the quantity $\frac{1}{2}I\omega^2$ is the final kinetic energy of the body; and the quantity $\frac{1}{2}I\omega_0^2$ is the initial kinetic energy of the body. In any given problem it is usually simpler to consider separately the work of the several forces acting upon the body rather than the work of their resultant.

If any of the forces are resistances, their torque is negative and their work is negative.

In any motion of rotation the work done by the positive forces minus the work done by the negative forces is equal to the increase in kinetic energy.

As in the case of translation, this may be written

Initial K.E. + positive work − negative work = final K.E.

EXAMPLE

A flywheel and shaft weighing 1800 lb. are rotating in bearings at 80 r.p.m. The shaft is 3 in. in diameter, and the radius of gyration of the shaft and

wheel is $k = 2.5$ ft. If the coefficient of friction $f = 0.01$, how long will the wheel rotate?

Solution.—The only force is friction, a resisting force. In one revolution its work is $fN \times 2\pi r$, so in n revolutions its work is $0.01 \times 1800 \times 2\pi \times 0.125n$, or $14.16n$ ft.-lb. The decrease in kinetic energy is the amount of the initial kinetic energy, or $\frac{1}{2}I\omega^2$.

$$I = Mk^2 = \frac{1800}{32.2} \times 6.25 = 349$$
$$\omega = \frac{80 \times 2\pi}{60} = 8.39$$
$$\omega^2 = 70.3$$
$$\tfrac{1}{2}I\omega^2 = 12{,}280 \text{ ft.-lb.}$$

Then

$$14.16n = 12{,}280$$
$$n = 867 \text{ rev.}$$

Since its acceleration is constant, the average r.p.m. = 40. The time required for 867 revolutions is given by $867 \div 40 = 21$ min. 40 sec.

Problems

1. The rotating cylinder of a brake-shoe testing machine is to be 4 ft. long and is to have the same kinetic energy as one-eighth of a 140,000-lb. railway car when the rim of the tested wheel has the same speed as the speed of the car. If it is to be made of cast iron, what must be its diameter? *Ans.* 3.7 ft.

2. If the speed of the rim of the wheel of the brake-shoe testing machine described in Prob. 1 is 60 m.p.h. when the brake is applied with a normal pressure of 15,000 lb., what is the coefficient of friction if the wheel travels 500 ft. under the brake? *Ans.* $f = 0.28$.

3. Compute the kinetic energy of the flywheel shown in Fig. 356 when it is rotating at 240 r.p.m. *Ans.* 34,000 ft.-lb.

4. A slender rod 6 ft. long weighing 30 lb. is free to rotate about a horizontal axis 1 ft. from the end. If it is released from rest with the center of gravity vertically above the support, compute the kinetic energy and the angular velocity after it has rotated 90°. After it has rotated 180°.
Ans. 60 ft.-lb., 4.29 rad./sec.; 120 ft.-lb., 6.07 rad./sec.

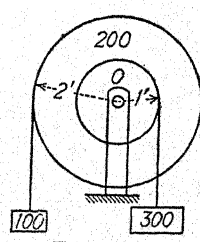


FIG. 444.

5. Figure 444 represents a pulley 2 ft. in diameter fastened to another 4 ft. in diameter and free to rotate about their common geometric axis O. The combined weight of the two pulleys is 200 lb., and their radius of gyration is 1.6 ft. If a weight of 300 lb. is hung from a cord wrapped around the smaller cylinder, and another of 100 lb. from a cord wrapped around the larger cylinder in the opposite direction, find the velocity with which the 300-lb. weight will strike the ground 4 ft. below the point where it is released from rest. Find also the distance that the 100-lb. weight will rise before coming to rest. *Ans.* $v = 4.61$ ft./sec.; $h = 3.01$ ft.

163. Kinetic Energy of Rotation and Translation.—A body that has a motion of combined rotation and translation may be considered at any instant to be rotating about its instantaneous axis. Its kinetic energy at that instant is given by the expression

$$\text{K.E.} = \tfrac{1}{2}I\omega^2$$

I being the moment of inertia of the body with respect to that instantaneous center and ω its angular velocity.

Since $I = I_g + M\bar{r}^2$ and $\bar{v} = \bar{r}\omega$, I_g being the moment of inertia of the body with respect to the gravity axis parallel to the instantaneous axis, $\bar{r}$ the distance between them, and $\bar{v}$ the absolute velocity of the center of gravity,

$$\begin{aligned}\text{K.E.} &= \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}I_g\omega^2 + \tfrac{1}{2}M\bar{r}^2\omega^2 \\ &= \tfrac{1}{2}I_g\omega^2 + \tfrac{1}{2}M\bar{v}^2\end{aligned}$$

The term $\tfrac{1}{2}M\bar{v}^2$ is the expression for the kinetic energy that the body would have if moving in translation with velocity $\bar{v}$, and the term $\tfrac{1}{2}I_g\omega^2$ is the kinetic energy that it would have if rotating about a fixed axis through the center of gravity parallel to the instantaneous axis with angular velocity ω. Hence the kinetic energy of a body with any plane motion is equal to the sum of the kinetic energies of translation and of rotation about the center of gravity.

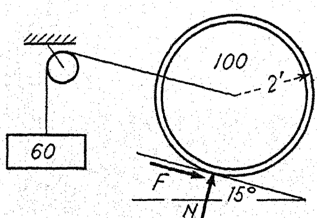


FIG. 445.

EXAMPLE

The wheel shown in Fig. 445 has a radius of gyration of 1.75 ft. and rolls freely on the plane. Get the velocity of the 60-lb. weight after it has moved 10 ft. from rest. Get the tension in the cord and the friction under the wheel.

Solution.—The initial kinetic energy is zero. The positive work is $60 \times 10 = 600$ ft.-lb. The negative work is $100 \times 0.259 \times 10 = 259$ ft.-lb. Forces F and N do no work, since their point of application is at rest. The final kinetic energy of the 60-lb. body is $\frac{1}{2}\frac{60}{32.2}v^2$. The final kinetic energy of the wheel is $\frac{1}{2}\frac{100}{32.2}v^2 + \frac{1}{2}\frac{100}{32.2}1.75^2\omega^2$. Since $v = 2\omega$, the equation of work and kinetic energy gives

$$\begin{aligned}600 - 259 &= \frac{1}{2}\frac{60}{32.2}v^2 + \frac{1}{2}\frac{100}{32.2}v^2 + \frac{1}{2}\frac{100}{32.2}\frac{1.75^2}{4}v^2 \\ v^2 &= 92.8 \\ v &= 9.63 \text{ ft./sec.}\end{aligned}$$

With the weight of 60 lb. as the free body, the equation of work and energy gives

$$(60 \times 10) - (T \times 10) = \left(\frac{1}{2}\frac{60}{32.2} \times 92.8\right)$$
$$T = 51.35 \text{ lb.}$$

Even though the frictional force F does no work on the cylinder, the center of the wheel may be used as the point of reference, and the kinetic energy of rotation of the wheel may be considered to be given to it by the force F acting through the distance that it moves with respect to the center.

$$10F = \frac{1}{2}\frac{100}{32.2}\frac{1.75^2}{4} \times 92.8$$
$$F = 11 \text{ lb.}$$

Problems

1. Get the expression for the kinetic energy of a solid circular cylinder rolling freely along a plane with a velocity v. *Ans.* $\frac{3}{4}Mv^2$.

2. Compute the kinetic energy of a hollow cast-iron sphere 10 in. outside diameter and 8 in. inside diameter that is rolling freely along a plane at a speed of 20 ft./sec. *Ans.* 644 ft.-lb.

3. A freight car weighing 100,000 lb. has four pairs of wheels such as those described in Prob. 1, Art. 116. Find the percentage of error if the rotational component of the kinetic energy of the wheels is neglected in computing the total kinetic energy of the car. *Ans.* 0.47 of 1 per cent.

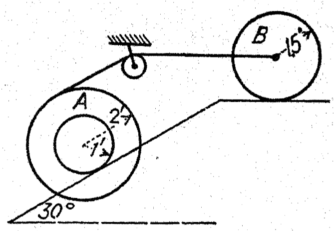


FIG. 446.

4. In Fig. 446, A is a solid circular cylinder 4 ft. in diameter and weighing 200 lb. The mass of the small cylinders on which it rolls may be neglected. The cord wrapped around cylinder A is connected to the axle of cylinder B which is 3 ft. in diameter and weighs 100 lb. Assuming free rolling at both places, get the linear and angular velocity of each cylinder after cylinder A has rolled 5 ft. from rest. Get also the friction under each cylinder and the tension in the cord.

Ans. Cylinder A, $v = 4.08$ ft./sec.; $\omega = 4.08$ rad./sec.; $F = 66.5$ lb. Cylinder B, $v = 12.24$ ft./sec.; $\omega = 8.16$ rad./sec.; $F = 7.7$ lb.; $T = 23$ lb.

164. Power and Efficiency.—*Power* is the rate of doing work, or the amount of work done per unit of time. If a weight of 100 pounds is lifted 10 feet, the work done is the same whether it is lifted in 1 second or in 5 seconds. The power required, however, is different. In the first case, it is 1000 foot-pounds per second, whereas in the second it is 200 foot-pounds per second.

The *unit of power* is the unit of work developed in the unit of time. In the English system it is, therefore, the *foot-pound per second* (abbreviated ft.-lb./sec.). This is too small a unit for

some engineering work, so the larger unit of *horsepower* is also used. The horsepower is 550 foot-pounds of work per second, or 33,000 foot-pounds of work per minute. In electrical work the unit of power commonly used is the *watt*, which is 10^7 ergs per second, or the *kilowatt*, which is 1000 watts. One horsepower = 0.746 kilowatt, or approximately $\frac{3}{4}$ kilowatt. If a force F moves through distance ds in dt time, its rate of doing work, or its power, is $F\frac{ds}{dt} = Fv$. So if F is in pounds and v in feet per second,

$$\text{Horsepower} = \frac{Fv}{550}$$

Because of friction and other resistances, such as air resistance, a certain amount of the energy supplied to a machine is lost, so the amount delivered by it, the output, is less than that delivered to it, the input. The ratio of the output to the input for a given length of time is called the *mechanical efficiency*.

$$\text{Efficiency} = \frac{\text{output}}{\text{input}}$$

Problems

1. If a hoisting engine lifts a mine cage weighing 1400 lb. a distance of 2000 ft. in 4 min., what horsepower is expended? If an indicator shows that 25.2 hp. is being developed, what is the efficiency of the engine and hoist? *Ans.* 21.21 hp.; 84.3 per cent.

2. Find the amount of useful work done by a pump that discharges 20 gal. of sulphuric acid per minute into a tank 60 ft. above the intake. Sulphuric acid weighs 112 lb./cu. ft. *Ans.* 17,970 ft.-lb./min., or 0.544 hp.

3. The driving side of a belt has 800 lb. tension, and the slack side has 350 lb. If the pulley is 6 ft. in diameter and has a speed of 200 r.p.m., what horsepower is being transmitted? If it is driving a dynamo that has an efficiency of 85 per cent, how many kilowatts are being delivered? *Ans.* 51.4 hp.; 32.6 kw.

4. An engine hoists 30 cu. ft. of concrete and a 400-lb. bucket a distance of 30 ft. in 16 sec. If concrete weighs 140 lb./cu. ft., what horsepower is the engine delivering? *Ans.* 15.7 hp.

165. Graphical Representation of Work.—Since work is the product of force and displacement, both of which are vector quantities, the graphical representation of work is made by means of an area. In Fig. 447(a), let AB represent the displacement s to some scale, and let AC represent the magnitude of the force to some scale. If the force is constant, the area $ABDC$ represents

to scale the work done, since it is the product of AB and AC. If AB represents 4 feet and AC represents 3 pounds, the area of each small rectangle represents 1 foot-pound of work, and the whole area represents 12 foot-pounds of work.

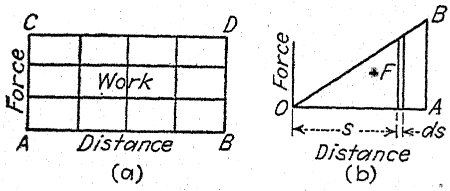


FIG. 447.

If the force varies in magnitude, the ordinates will not be the same height, but the area will still give the work done. By calculus, the area under the curve that has abscissæ s and ordinates F is equal to $\int F\,ds$. The simplest case is that in which the force varies as the distance, increasing from zero to a maximum or decreasing from an initial maximum to zero. The diagram for this case is a triangle, Fig. 447(b), in which the ordinate AB represents to some scale the maximum value of the force F, which has increased uniformly from zero. The work done is represented by the triangular area OBA, which is equal to $\frac{1}{2}AB \times OA$. In terms of the maximum force F_1 and the total distance s_1,

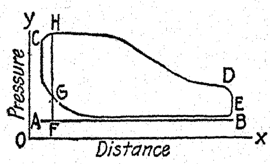


FIG. 448.

$$\text{Work} = \frac{1}{2}F_1 s_1$$

The steam-engine indicator is an instrument for recording graphically the steam pressure and piston travel of a steam engine. In Fig. 448, OX is the axis of zero absolute pressure. AB is the line showing atmospheric pressure, and its length represents to some scale the piston travel. The ordinate FH represents to some scale the steam pressure when the piston was at the corresponding point on its working stroke, and FG represents the back pressure at the same point on the return stroke. The area $GCHDE$ represents to scale the work done by the steam. This area may be computed approximately by Simpson's one-third rule or may be measured with a planimeter.

If the area of the indicator diagram is divided by the length AB, the result will be the average ordinate, which multiplied by the scale of the spring will give the *mean effective pressure.* This is the pressure which, if exerted through the whole stroke, would have done the same amount of work.

Let P represent the mean effective pressure in pounds per square inch, l the length of the cylinder in feet, a the area of the piston in square inches, and n the number of revolutions per minute made by the flywheel. Then Pa is the total pressure in pounds and Pla is the work done in one revolution by the steam on one side of the piston. $Plan$ is the work done in 1 minute and $Plan/33{,}000$ is the horsepower generated. This is called the *indicated horsepower.* If the piston rod extends both ways from the piston so that the areas are the same and the mean effective pressures are the same, the total power generated in the cylinder is $2\ Plan/33{,}000$. If, as is commonly the case, the pressures and areas are different, the horsepowers of the two ends are computed separately and added.

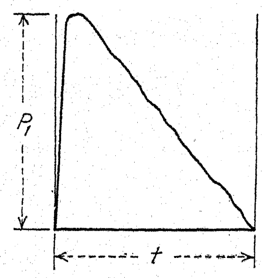


FIG. 449.

In punching a hole through a plate, the pressure on the punch increases rapidly from zero to a maximum value P_1, Fig. 449; then decreases to zero with approximately straight-line variation. The area of the work diagram is approximately $\frac{1}{2}P_1t$, t being the thickness of the plate.

Problems

1. A force is applied to a 16,000-lb. spring to compress it $3\frac{1}{2}$ in., then released $\frac{1}{2}$ in. Draw the work diagram, and from it compute the gross work, the negative work, and the net work.

Ans. 98,000 in.-lb.; 26,000 in.-lb.; 72,000 in.-lb.

2. Compute the work done in punching an $1\frac{1}{16}$-in. hole through a $\frac{1}{2}$-in. plate for which the value of the ultimate unit shearing stress is 44,000 lb./in.^{2}. *Ans.* 11,880 in.-lb.

3. The mean effective pressure of the steam in the cylinder of an engine is 80 lb./in.^{2}; the crank is 12 in. long; the cylinder is 10 in. in diameter; the piston rod is 1.5 in. in diameter and extends entirely through the cylinder. If the engine is running at 210 r.p.m., what is the indicated horsepower?

Ans. 156 hp.

4. At the crank end of the cylinder of an engine, the mean effective pressure of the steam is 98 lb./in.^{2}, and at the head end it is 95 lb./in.^{2}. The length of the crank is 1 ft.; the diameter of the cylinder is 1 ft.; and the

diameter of the piston rod is 2 in. The piston rod extends only one way from the piston. If the speed of the engine is 90 r.p.m., what horsepower is it developing?

Ans. 117.5 hp.

166. Work Lost in Friction.—Friction is the great reducing agent by means of which kinetic energy is dissipated in the form of heat. When two surfaces move over each other, the mutual friction reduces the total kinetic energy by an amount equal to the product of the frictional force and the relative motion. Let the crosshead of a locomotive have an average pressure on its guides of 500 pounds, and let the coefficient of friction $f = 0.01$. If the crank is 1 foot long, the relative distance that the crosshead moves over the guides during the forward stroke is 2 feet. The work lost in friction between the crosshead and guides is then $0.01 \times 500 \times 2 = 10$ foot-pounds.

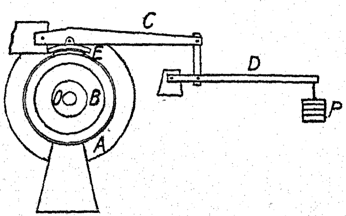


FIG. 450.

This cannot be recovered in the form of useful work in another part of the motion, as in the case of the work required for accelerating the piston, for the friction changes direction, and the same amount more is lost during the return stroke.

The brake-shoe testing machine (Fig. 450) consists of a heavy drum A rigidly fastened to an axle O, to which is also fastened a car wheel B. By means of the levers C and D, the weight P applies a normal pressure to the rim of the wheel through the brake shoe E. The weights and dimensions are known, so the kinetic energy may be computed when the angular velocity is known. The weight of the rotating drum is 12,600 pounds, practically the same as one-eighth of a 100,000-pound car, or the part supported by one wheel. The material is so arranged that its rotary kinetic energy is the same as the translatory kinetic energy of the same weight would be if it had the same speed as the rim of the wheel. The requirement for this is that $\frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2$. Since $I = Mk^2$ and $v = r\omega$, it is necessary that $k = r$, so the drum is built in such a way as to make $k = 1.375$ feet, the radius of the standard car wheel. The brake shoe on the testing machine is under the same conditions as in service.

The axle is connected to an engine by means of a clutch so that any desired speed may be given to it, after which the engine may

be disconnected. The drum and wheel will then rotate freely until the weight of P is applied which presses the brake shoe E against the wheel. The arrangement of the levers is such that a weight P at the end of the lever produces a normal pressure of 24 P at E. In addition, the weight of the levers themselves produces a normal pressure of 1230 pounds. Let the total normal force be N. Then the frictional drag of the shoe upon the wheel is fN, f being the kinetic coefficient of friction, and the work of the frictional force is $fN\pi{}^{33}\!/_{12}$ foot-pounds in one revolution.

Some work is done also by the frictional force at the bearings. The pressure here is $12{,}600 + N$. The axle is 7 inches in diameter; and if f_1 is the coefficient of friction, the work lost in one revolution will be $f_1(12{,}600 + N)\pi{}^7\!/_{12}$ foot-pounds. The work of friction at the two points dissipates the kinetic energy of the drum and wheel. That at the brake shoe is, of course, much the larger. The kinetic energy is transformed into heat, sometimes making the surface of the shoe red hot.

If n is the number of revolutions of the wheel, the work-energy equation for the motion becomes

$$K.E. = \text{work of friction.}$$

$$\tfrac{1}{2}I\omega^2 = fN\pi n \times {}^{33}\!/_{12} + f_1(12{,}600 + N)\pi n \times {}^7\!/_{12}$$

Problems

1. Use $f = 0.3$ and $f_1 = 0.004$ for the brake-shoe testing machine described above. If the rim speed is 60 m.p.h., what weight P will bring the wheel to rest in 120 revolutions? *Ans.* 150 lb.

2. If the rim speed of the wheel on the testing machine is 75 m.p.h. and with 300 lb. at P is brought to rest in 125 revolutions, what is the value of f? Use the same value for f_1 as in Prob. 1. *Ans.* $f = 0.258$.

3. If for the brake-shoe testing machine the coefficient of friction $f = 0.25$, $f_1 = 0.005$, and the force $P = 500$ lb., in what distance will the wheel be stopped when running with a rim speed of 90 m.p.h.? *Ans.* 1022 ft.

167. Braking of Trains.—A train running at a high rate of speed has a large amount of kinetic energy which has been given to it by the work of the steam in the cylinders or, if on a down grade, by gravity also. When the train is to be stopped, all this kinetic energy must be used up again in work. The usual method is to press brake shoes against the rims of the wheels and so transform the kinetic energy of the train into heat at the rubbing surfaces. The force of friction does work of retardation

equal to fNs, f being the coefficient of friction, N the normal pressure of the brake shoe on the wheel, and s the distance traveled by the rim of the wheel relative to the brake. The action is a tendency to check the rotation of the wheel, so that a backward static frictional force is developed at the point of contact of the wheel and the rail. This force of the rail on the wheel is the one that actually stops the train, but it does no work since its point of application does not move in the direction of the force. The maximum braking force is exerted when skidding of the wheel is impending, but the wheel is still rolling, as in Fig. 451(a), for then the limiting or maximum value of the static friction at the rail is induced. If the friction at the axle is neglected, the equa-

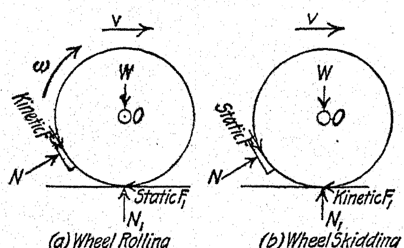


FIG. 451.

tion of moments about the axle shows that the two frictional forces F_1 and F are equal when the wheel is under static conditions.

If, now, the normal force on the brake shoe is slightly increased, the frictional force at the shoe will become slightly greater and the wheel will skid, as in Fig. 451(b). There is now kinetic friction between the wheel and the rail which is less than static friction, so resistance to motion is less. The work is being done at the point of contact of the wheel and the rail instead of at the surface of the brake shoe as before, whereas the brake shoe does no work and is not worn. Skidding of the wheels, besides being much less efficient in stopping the train, causes injurious flat spots to be worn on the wheel.

If a locomotive is moving forward, and it is reversed so that the driving wheels rotate backward, the friction is kinetic friction which is less than static friction. The effect in stopping the locomotive is therefore less than if the wheels are rotating forward with skidding impending, as can readily be seen from the relation $F = Ma$. The work of friction may be larger, however, in any given horizontal distance s, since the distance traveled by the

frictional force is $s + s'$, s' being the circumferential distance traveled by a point on the rim with respect to the center of the wheel. The work Fs reduces the kinetic energy of the locomotive and train. The work Fs' balances the work of the steam in the cylinders. If sand is applied to the rails, either with the brakes applied or with the locomotive reversed, an additional resisting force is developed due to the abrasion.

The coefficient of friction between brake shoes and car wheels is extremely variable on account of the following factors: material of the shoe, material of the wheel, initial speed of the train, speed of the wheel at the time considered, and weather conditions. Just before the wheel is stopped, the coefficient is considerably higher than the average for the stop. At high speeds the coefficient is much less than at low speeds, because of the heating of the material at the rubbing surfaces.

The following table gives average results from some M. C. B. Association tests upon several different kinds of brake shoes, at two different speeds:

Speed, miles per hour	Average f	Final f
40	0.205	0.326
65	0.103	0.180

It is seen that both the average and the final values of f when the initial speed is 65 miles per hour are much less than they are when the initial speed is 40 miles per hour. This is due to the fact that at 65 miles per hour there is more than two and one-half times as much kinetic energy to be used up at the rubbing surfaces. The surfaces are heated more and their gripping qualities lessened. When stopping trains at high speeds, it is customary to apply the brakes at first with a heavy pressure until the speed is partially reduced, then to release and apply again with less pressure in order to avoid skidding as the train comes to rest.

Since the average value of the final kinetic coefficient of friction for low speeds is as high as the value of the static coefficient of friction between the wheel and the rail, the normal pressure on the brake shoe should not be greater than the minimum weight carried by the wheel if skidding of the wheel is to be avoided. When the car is loaded, the braking effect is necessarily very much less than its maximum value.

Problems

1. If a freight car weighing 40,000 lb. when empty has a normal brake-shoe pressure of 5000 lb. on each of its eight wheels, what is the shortest possible distance in which the car could be brought to rest from a speed of 45 m.p.h.? Assume kinetic $f = 0.22$ and that it is constant and also that the wheels do not skid on the track. Ans. 308 ft.

2. If the car of Prob. 1 carries a load of 100,000 lb., what is the shortest possible distance in which it could be brought to rest from a speed of 45 m.p.h., assuming the same value of the coefficient of friction? *Ans.* 1078 ft.

3. A 3000-ton train while running at a speed of 40 m.p.h. down a 0.5 per cent grade has brakes applied so that it is brought to rest in a distance of 1600 ft. Compute the total induced resisting force required, assuming train resistance as constant at 7 lb./ton. *Ans.* 209,400 lb.

168. Flexible Band Brakes.—Figure 452 shows a simple form of band brake, such as is used on hoisting engines. The band is attached at C and passes around the wheel to the lever at B. The lever is hinged at A; if force is applied downward at the end, the band is tightened, and the friction retards the motion of the wheel inside the band. If the weight W is to be lowered at a uniform rate of speed, the work done by gravity upon the weight must equal the work done by the frictional force $T_2 - T_1$ on the rim of the brake wheel, or

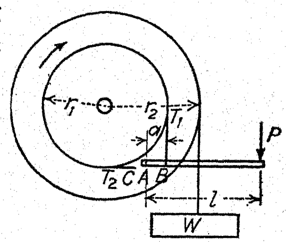


FIG. 452.

$$2\pi r_2 W = (T_2 - T_1)2\pi r_1$$

From Art. 68,

$$T_2 = T_1 e^{f\beta}$$

f being the kinetic coefficient of friction, and β the angle of contact. By moments about point A, with the lever as the free body,

$$Pl = T_1 a$$

From these three equations, the force P required to lower the weight W may be determined.

Problems

1. If in Fig. 452 $r_1 = 2$ ft., $r_2 = 3$ ft., $W = 2000$ lb., $a = 10$ in., $l = 8$ ft., and f [illegible] what force P is required in order that the weight may descend at con[illegible]speed? *Ans.* 200 lb.

2. If $P = 180$ lb. on the band brake of Prob. 1, with what velocity will the weight W pass a point 40 ft. below the starting point? The wheel and drum weigh 1200 lb., and their radius of gyration is 2.4 ft. *Ans.* 13.64 ft./sec.

3. With the same data as in Prob. 1, but with the cable supporting W wrapped the other way around the drum, what force P will be required so that the weight may descend at constant speed? *Ans.* 512 lb.

169. Absorption Dynamometer.—An absorption dynamometer is an instrument for measuring the output of power of such machines as steam engines, electric motors, and water wheels. It absorbs all the energy generated and transforms it into heat of friction. The most common form of absorption dynamometer

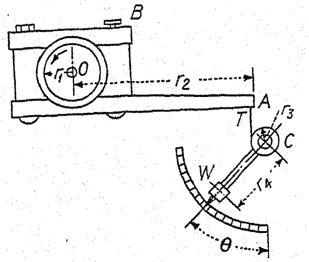


FIG. 453.

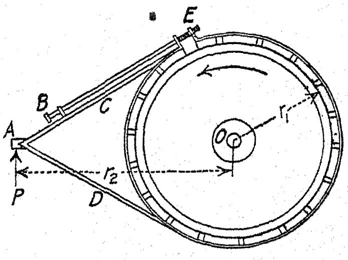


FIG. 454.

is the Prony brake (Figs. 453 and 454). The simple construction of Fig. 453 is best for small, high-speed machines, as motors and small gas engines. By tightening the hand wheel B, the blocks of which the brake is composed are clamped against the wheel, and the friction developed tends to turn the brake around with the wheel. This tendency is resisted by the pull of the weight W, hinged at C. The hand wheel B is tightened until all the work done by the prime mover is used up in the heat of friction at the rim.

Let F be the total friction generated. Then the work absorbed in one revolution is $F \times 2\pi r_1$; if n is the number of revolutions per minute, the horsepower is given by

$$\text{Hp.} = \frac{F \times 2\pi r_1 n}{33{,}000}$$

But $Fr_1 = Tr_2$, by moments about O; and $Tr_3 = Wr_4 \sin\theta$, by moments about C, so

$$\text{Hp.} = \frac{2\pi n W r_2 r_4 \sin\theta}{33{,}000 r_3}$$

Instead of being graduated in degrees, the arc may be graduated to read T directly or, more simply, the frictional force F.

The brake shown in Fig. 454 is used for larger sizes of flywheels. It is composed of small blocks of wood fastened to a strap encircling the wheel and attached to a V-shaped lever arm CDA. To apply the brake, the ends of the band at E are drawn together by turning the hand wheel B. As before,

$$Fr_1 = Pr_2$$

so

$$\text{Hp.} = \frac{2\pi nPr_2}{33{,}000}$$

The force P is the net force due to friction alone after the weight of the brake arm has been balanced. It is usually measured by resting the lever arm on a platform scale or by suspending it from a spring scale above.

Since some of the work done by the steam in the cylinder is used up in friction of the moving parts of the engine, the brake horsepower will necessarily be less than the indicated horsepower. The mechanical efficiency is the ratio of the brake horsepower to the indicated horsepower.

Problems

1. In a brake of the kind shown in Fig. 453, $r_1 = 4.5$ in., $r_2 = 3$ ft., $r_3 = 2$ in., $r_4 = 15$ in., and $W = 10$ lb. If the brake is balanced so that the pointer is vertical with no load, at what angles should the calibration marks be placed for $F = 100$ lb., 200 lb., 300 lb., and 400 lb.?
Ans. 9°36′; 19°28′; 30°00′; 41°49′.

2. If a motor being tested by the brake described in Prob. 1 is running at 1800 r.p.m., and the pointer reads 362 lb., what is the brake horsepower?
Ans. 46.5 hp.

3. A brake of the style of Fig. 454 has the radius $r_1 = 2$ ft. and $r_2 = 6$ ft. If when a certain engine is being tested $P = 380$ lb. and $n = 210$ r.p.m., what horsepower is being developed? *Ans.* 91.2 hp.

4. If the engine referred to in Prob. 3 has a cylinder 16 in. in diameter, piston rod 3 in. in diameter (extending only one way from the piston), 24-in. stroke, and mean effective pressure of 22 lb./in.^{2} at the head end and 20 lb./in.^{2} at the crank end, what is the mechanical efficiency of the engine?
Ans. 86.4 per cent.

170. Water Power.—A stream or jet of water has kinetic energy due to its motion and is capable of doing work by giving up some of this kinetic energy to a machine as it passes through it. Let W be the number of pounds of water flowing past a given point of a stream per second, and v its velocity in feet per second. Then the kinetic energy of the stream per second is $\frac{1}{2}Wv^2/g$. If all this kinetic energy could be destroyed in doing

useful work, the horsepower available would be

$$\text{Hp.} = \frac{1}{2}\frac{W}{g}\frac{v^2}{550}$$

If a stream of water is issuing from a nozzle with velocity v under head h, the relation between v and h (neglecting losses) is given by the expression

$$v^2 = 2gh$$

$$\text{Hp.} = \frac{Wh}{550}$$

Problems

1. Water issues from a 3-in. nozzle under a head of 960 ft. against the blades of an impulse turbine with an efficiency of 85 per cent. The turbine is connected to generators that have an efficiency of 90 per cent. How many kilowatts are delivered at the switchboard? *Ans.* 760 kw.

2. If a volume of 840 cu. ft. of water per second under a head of 60 ft. flows through turbines that have an efficiency of 82 per cent, what horsepower will they deliver? *Ans.* 4700 hp.

GENERAL PROBLEMS ON WORK, ENERGY, AND POWER

1. Compute the amount of work done in elevating the clay from a pit 25 ft. in diameter and 60 ft. deep if the clay weighs 110 lb./cu. ft. and is lifted 8 ft. above the top of the pit. *Ans.* 61,580 ft.-tons.

2. An elevated cylindrical water tank is 24 ft. in diameter and 20 ft. high. It has a conical bottom on a 45° slope and a riser 1 ft. in diameter and 60 ft. high. If water is drawn from deep wells with an average level of 100 ft. below the base of the riser, what work is done against gravity in filling the riser and tank? If the pumps have an efficiency of 85 per cent, and the filling is done in 10 hours, no water being drawn out meanwhile, what horsepower is required? *Ans.* 122,400,000 ft.-lb.; 7.25 hp.

3. A fire engine takes water from the surface of a lake 16 ft. below its own level and delivers it from a nozzle 2 in. in diameter with a velocity of 240 ft./sec. What horsepower is required? *Ans.* 542 hp.

4. A tank 8 ft. long, 6 ft. wide, and 4 ft. deep is two-thirds full of water. How many foot pounds of work are required to raise all the water 6 in. above the top of the tank? *Ans.* 25,330 ft.-lb.

5. A weight of 2000 lb. hangs from a winding drum by a cable 800 ft. long, weighing 1 lb./lin. ft. How much work is done as the weight is raised a distance of 600 ft.? *Ans.* 1,500,000 ft.-lb.

6. Figure 455 represents a solid circular cylinder 1 ft. in diameter and weighing 240 lb., free to roll on a 30° plane. Suspended from the cord connected to the axis of the cylinder is a weight of 100 lb. Get the linear and angular velocity of the cylinder after it has rolled 2 ft. from rest, assuming free rolling. Get the tension in the cord and the friction of the plane on the cylinder.

Ans. $v = 3.033$ ft./sec.; $\omega = 6.066$ rad./sec.; $T = 107.15$ lb.; $F = 8.57$ lb.

7. If the cylinder and block referred to in Prob. 6 are arranged on the 30° plane as shown in Fig. 456, compute the linear and angular velocity of the cylinder, the tension in the cord, and the friction under the cylinder. Static $f = 0.24$; kinetic $f = 0.20$.

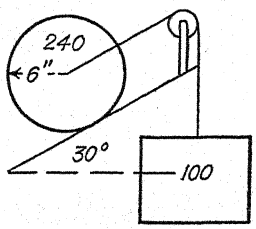


FIG. 455.

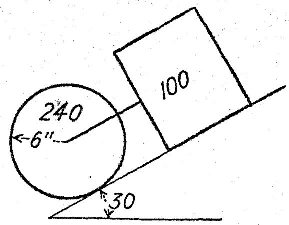


FIG. 456.

Ans. $v = 6.54$ ft./sec.; $\omega = 13.08$ rad./sec.; $T = 0.52$ lb.; $F = 39.8$ lb.

8. Figure 457 represents a cast-iron disk 4 ft. in diameter and 3 in. thick with a concentric disk 8 in. in diameter and 3 in. thick on each side upon which it is free to roll on the inclined plane. Get the amount W of the counterweight necessary to give the disk a linear velocity of 5 ft./sec. up the plane in a distance of 10 ft. from rest. Get also the tension in the cord and the friction under the disk. Assume free rolling.

Ans. $W = 1492$ lb.; $T = 1434$ lb.; $F = 990$ lb.

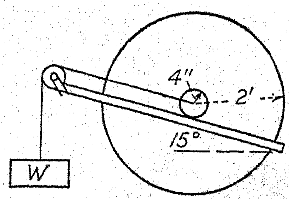


FIG. 457.

9. If after moving 10 ft. the counterweight W, Fig. 457, is disconnected, how much farther up the plane will the disk roll? What is the amount of the frictional force F during this motion? *Ans.* $s = 27.2$ ft.; $F = 364$ lb.

10. If after coming to rest the disk of Prob. 9 is allowed to roll back down the incline, with what linear velocity will it reach the original starting point? *Ans.* $v = 5.82$ ft./sec.

11. A weight of 100 lb. rests upon a vertical coil spring, the scale of which is 800 lb./in. The spring is then pressed down 6 in. farther by an added pressure of 4800 lb. If this added pressure is then released suddenly, what will be the velocity of the 100-lb. weight 5 ft. above the depressed position? How much higher will it rise? *Ans.* $v = 22$ ft./sec.; 7.5 ft.

12. If in a steam hammer of the same dimensions as given in Prob. 4, Art. 161, the steam is cut off and released at the top end of the cylinder when the piston is at quarter stroke and at the same time is admitted at boiler pressure below the piston, how far down will the hammer move? *Ans.* 1.735 ft.

13. In the hammer described in Prob. 12, at what point must cutoff and release above, and admission below, be made in order that the hammer may just touch the anvil? *Ans.* 1.3 ft.

14. If in the hammer described in Prob. 12 the cutoff is made at quarter stroke, but release above and admission below is made at half stroke, with what velocity will the hammer strike the anvil? *Ans.* 8.02 ft./sec.

15. The steam-indicator card for the head end of the cylinder of a steam engine had an area of 3.27 sq. in., and for the crank end, 3.21 sq. in. Length of atmosphere line was 3.25 in.; scale of the spring, 40 lb. per in.; length of stroke, 24 in.; diameter of piston, 10 in.; diameter of piston rod, 1.25 in. If the engine is running at 150 r.p.m., what is the indicated horsepower? *Ans.* 56.5 hp.

16. An 800-ton train attains a speed of 40 m.p.h. in 1 mile on a level track with a constant drawbar pull. If train resistance is considered constant and equal to 8 lb./ton, what is the necessary drawbar pull? What is the maximum horsepower? *Ans.* 22,600 lb.; 2410 hp.

17. If the locomotive of Prob. 16 is pulling the same train up a 0.5 per cent grade and is exerting the same drawbar pull, in what distance will it attain a speed of 40 m.p.h.? *Ans.* 10,430 ft.

18. A freight car weighing 120,000 lb. is running down a 2 per cent grade with a speed of 15 m.p.h. If train resistance is 9 lb./ton, what normal brake-shoe pressure is necessary on each of the eight wheels in order to stop the car in a distance of 1200 ft.? Use kinetic $f = 0.25$ between brake shoe and wheel. *Ans.* $N = 1305$ lb.

19. A 300-ton train is running at a speed of 90 m.p.h. on a level track. If train resistance is 20 lb./ton, what is the drawbar pull? What horsepower is the locomotive developing? *Ans.* 6000 lb.; 1440 hp.

20. A Prony brake on a 4-ft. flywheel has a lever arm 6 ft. long. When the wheel is rotating at a speed of 210 r.p.m., and the scale under the end of the lever reads 455 lb., what horsepower is being developed? *Ans.* 109 hp.

21. A hoisting engine is lifting 1 ton of ore per minute from a steamer's hold 60 ft. deep. If the efficiency of the hoisting apparatus is 70 per cent, and that of the engine is 85 per cent, what is the indicated horsepower? *Ans.* 6.11 hp.

22. What horsepower is being developed if an 800-ton train is being pulled up a 0.4 per cent grade at a speed of 45 m.p.h., the train resistance being 12 lb./ton? *Ans.* 1920 hp.

23. The light weight of the train described in Prob. 22 is 200 tons, and it is equipped with brakes to give a normal pressure of nine-tenths of the light weight. Assuming $f = 0.25$, compute the shortest distance in which the train when loaded could be brought to rest from a speed of 45 m.p.h. on a level track. Assume train resistance constant at 10 lb./ton. *Ans.* 1104 ft.

24. A Pelton wheel is driven by a jet of water 1 in. in diameter under a head of 350 ft. If the efficiency of the wheel is 85 per cent, what horsepower will be generated? If the wheel is directly connected to a generator that has 88 per cent efficiency, how many kilowatts will be delivered? *Ans.* 27.6 hp.; 18.2 kw.

25. Compute the kinetic energy of a racing car weighing 8000 lb. when traveling at a speed of 345 m.p.h. If when traveling at a speed of 345 m.p.h. the car is brought to rest in a distance of 5 miles, what is the average resisting force developed? *Ans.* 31,800,000 ft.-lb.; 1200 lb.

CHAPTER XVII

IMPULSE, MOMENTUM, AND IMPACT

171. Impulse and Momentum.—The effect of a force may be given in terms of the product of *force* and *distance*, which is called *work;* or the product of *force* and *time*, which is called the *impulse* of the force. If a force F is constant in both magnitude and direction during time t, the impulse is Ft. If F varies in magnitude, the impulse for the infinitesimal time dt is $F\,dt$, and the impulse for any time t is given by $\int_0^t F\,dt$. If the relation of F and t is known, the integration may be performed. Sometimes, even though the relation between F and t is not known, the quantity $\int_0^t F\,dt$ can be eliminated between two simultaneous equations containing it.

Impulse, like force, is a *vector* quantity and has the same direction and position as the force factor.

The resultant impulse of a force that varies in direction is the vector sum of the separate component impulses. Consider a resultant force F applied to a body for t seconds, then suddenly reversed and applied for a succeeding t seconds. It is evident that the vectorial sum of the impulses is zero, since they are of the same numerical value and of opposite sign.

If in the preceding case the force should be changed only through 90°, the resultant impulse would be $Ft\sqrt{2}$ in amount, at an angle of 45° with the direction of either component impulse. In any case in which the force varies in direction, $\int F\,dt$ must be vectorial.

The *unit of impulse* is the impulse of a unit force acting for a unit of time. In the English system this is the *pound-second.*

The *momentum* of a body is the product of its mass and velocity Mv. It is sometimes called the *quantity of motion.* Momentum, like velocity, is a vector quantity having definite direction and position. Like other vector quantities, both impulse and momentum may be resolved into components or combined into resultants.

The *unit of momentum* is the momentum of a unit mass moving with unit velocity. In the English system the dimensions of this unit are obtained as follows: Since $M = W/g$, W being in pounds and g in $\frac{\text{feet}}{\text{seconds}^2}$, M is in units of $\frac{\text{pounds} \times \text{seconds}^2}{\text{feet}}$. Velocity v is in units of feet/seconds, so Mv is in units of

$$\frac{\text{pounds} \times \text{seconds}^2}{\text{feet}} \times \frac{\text{feet}}{\text{seconds}} = \text{pounds} \times \text{seconds}.$$

The dimensions of the unit of momentum are, therefore, the same as those of the unit of impulse.

Problems

1. The initial value of a force is 4 lb., and it increases with the time at the rate of 0.1 lb./sec. What is the impulse of the force during the first 10 sec.? What is the impulse of the force during the second 10 sec.?
Ans. 45 lb.-sec.; 55 lb.-sec.

2. The initial value of a force is 6 lb., and it decreases with the time at the rate of 0.5 lb./sec. What is the impulse of the force during the first 10 sec.? *Ans.* 35 lb.-sec.

3. Compute the momentum of an 8000-lb. racing car traveling at a speed of 345 m.p.h. *Ans.* 125,600 lb.-sec.

172. Relation between Impulse and Momentum.—Let F be the resultant force acting upon a body of mass M to produce an acceleration a. Then $F = Ma$. Since

$$a = \frac{dv}{dt}$$

$$F = M\frac{dv}{dt}$$

$$F\,dt = M\,dv$$

Let the limits of t be 0 and t and the corresponding velocities be v_0 and v.

$$\int_0^t F\,dt = \int_{v_0}^v M\,dv$$

If the mass M is assumed to be constant,

$$\int_0^t \mathbf{F}\,\mathbf{dt} = \mathbf{Mv} - \mathbf{Mv_0}$$

If the mass M is not constant, the relation of M to v must be known before the expression can be integrated.

If both the mass M and the force F are constant,

$$\mathbf{Ft} = \mathbf{Mv} - \mathbf{Mv}_0$$

The general statement of this relation may be made as follows: During any period of time t the impulse of the resultant force acting upon a body is equal to its change in momentum.

By means of the foregoing relation, problems involving force, mass, velocity, and time may be solved directly instead of with the double set of equations between force, mass, and acceleration, and velocity, acceleration, and time.

EXAMPLE 1

A body is thrown vertically upward with an initial velocity of 30 ft./sec. Assuming that it can fall freely after it comes to rest at the top point in its path, find its velocity 2.5 sec. after discharge.

Solution.—The resultant force F is the weight W and is negative, since it is in the direction opposite to the initial velocity. The impulse of the force is equal to the change in momentum.

$$Ft = Mv - Mv_0$$

$$-2.5W = \frac{W}{32.2}v - \frac{W \times 30}{32.2}$$

$$v = -50.5 \text{ ft./sec.}$$

The negative sign shows that the velocity is downward.

EXAMPLE 2

A 500-lb. body initially at rest is acted upon for 10 sec. by a variable working force F which is equal to $100\sqrt{t}$ and also by a variable resisting frictional force F_1 which is approximately equal to $20 - t$ during that time. What is its velocity at the end of 10 sec.?

Solution.

$$\text{Impulse} = \text{change in momentum}$$

$$\int_0^{10} 100t^{1/2}\, dt - \int_0^{10} 20\, dt + \int_0^{10} t\, dt = \frac{500}{32.2}v$$

$$\frac{200}{3}t^{3/2}\Big]_0^{10} - 20t\Big]_0^{10} + \frac{t^2}{2}\Big]_0^{10} = \frac{500}{32.2}v$$

$$v = 126 \text{ ft./sec.}$$

Problems

1. A body is thrown vertically downward with a velocity of 20 ft./sec. What is its velocity after 4 sec.? *Ans.* 148.8 ft./sec.

2. If a 120-lb. weight on a level floor has a horizontal force of 15 lb. acting upon it for 6 sec. and a frictional force equal to $10 - t^{1/3}$ during that time, what will be its velocity if it starts from rest? *Ans.* $v = 10.24$ ft./sec.

3. Solve Prob. 2 if the weight has an initial velocity to the left of 10 ft./sec. and the 15-lb. force is acting toward the right?

Ans. $v = 7.8$ ft./sec.

173. Conservation of Linear Momentum.—In any mutual action between two bodies or two parts of the same body, the mutual forces are always equal and opposite, by Newton's third law of motion. Since the time of contact is necessarily the same, the impulses of the mutual forces are equal in value and opposite in direction, hence neutralize each other. Then if there is no resultant force acting which is external to the two bodies, the sum of the momenta before the action is equal to the sum of the momenta after the action, since for the whole system $\int F\,dt = 0$. If M_1 and M_2 are the masses of two bodies, v_1 and v_2 their velocities before contact, and v_1' and v_2' their velocities after contact, then

$$M_1v_1 + M_2v_2 = M_1v_1' + M_2v_2'$$

Though there is no loss of momentum in any mutual action between two bodies, there is always a loss in kinetic energy due to the heat generated at the point of contact.

The direction of v_1 should always be considered positive. If v_2 is in the opposite direction, it must be used as negative.

EXAMPLE

A 50-lb. shot is fired from a gun that weighs 20,000 lb. If its muzzle velocity is 1200 ft./sec., what is the initial backward velocity of the gun? If the recoil is against a constant force of 3000 lb., how soon will the gun be brought to rest? What distance does it recoil? What is the kinetic energy of each?

Solution.—The momentum of the shot and the gun before the shot is fired is equal to zero, so the sum of the momenta after the shot is fired must also equal zero. Therefore,

Momentum of shot forward − momentum of gun backward = 0

$$\frac{50}{32.2} \times 1200 = \frac{20{,}000}{32.2}v$$

$$v = 3 \text{ ft./sec.}$$

The impulse of the resisting force is equal to the momentum of the gun, so

$$\frac{20{,}000}{32.2} \times 3 = 3000t$$

$$t = 0.621 \text{ sec.}$$

Since the resisting force is constant, the motion is one with uniform acceleration, and the distance is equal to the product of the average velocity and the time. The average velocity is $\frac{1}{2} \times 3 = 1.5$, so

$$s = 1.5 \times 0.621 = 0.932 \text{ ft.}$$

The kinetic energy of the shot is

$$\frac{1}{2} \times \frac{50}{32.2} \times 1200^2 = 1{,}120{,}000 \text{ ft.-lb.}$$

The kinetic energy of the gun is

$$\frac{1}{2} \times \frac{20{,}000}{32.2} \times 3^2 = 2790 \text{ ft.-lb.}$$

It will be noticed that though the momentum of the shot is numerically equal to that of the gun, its kinetic energy is four hundred times as great.

Problems

1. A man weighing 170 lb. jumps with a horizontal velocity of 10 ft./sec. into a boat that is at rest on the water. If the boat weighs 200 lb., what is its velocity when the man comes to rest with respect to it? What is the loss in kinetic energy? *Ans.* 4.6 ft./sec.; 142 ft.-lb.

2. A bullet weighing $\frac{1}{6}$ oz. is shot horizontally into a block weighing 3 lb. which is at rest on a smooth horizontal surface. If the velocity of the block when the bullet has come to rest in it is 12 ft./sec., what was the muzzle velocity of the bullet? Compute the loss in kinetic energy.

Ans. 3470 ft./sec.; 1938 ft.-lb.

3. A gun weighing 160,000 lb. fires a projectile weighing 600 lb. with a velocity of 2000 ft./sec. With what initial velocity will the gun recoil? How far will it recoil if resisted by a constant force of 90,000 lb.? Get the initial kinetic energy of the projectile and of the gun.

Ans. 7.5 ft./sec.; 18.6 in.; 37,300,000 ft.-lb.; 140,000 ft.-lb.

174. Angular Impulse and Angular Momentum.—Let Fig. 458 represent a body free to rotate about a fixed axis O. Let F be the resultant of all the rotating forces in the plane of motion of the body and let d be its moment arm with respect to the axis O. The impulse of the force F during time dt is $F\,dt$, and this impulse is a vector quantity having direction and line of action as shown. The moment of this impulse about the axis O is $d \times F\,dt$, and the summation of these moments of impulse $\int d \times F\,dt$ is called the *moment of impulse,* or *angular impulse,* of the force upon the body.

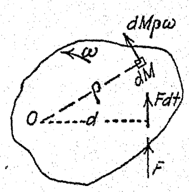


FIG. 458

Let dM be the mass of any particle at distance ρ from the axis of revolution O. If the angular velocity of the body is ω, the tangential velocity of the mass dM is $\rho\omega$ and its momentum is $dM\rho\omega$. This momentum is a vector quantity whose position is through dM in the direction of the velocity of dM, normal

to ρ. The moment of the elementary momentum about the axis O is $dM\rho^2\omega$ and is called the moment of momentum of the particle. The summation of all these elementary moments of momentum is called the *moment of momentum*, or *angular momentum*, of the body.

$$\int dM\rho^2\omega = \omega\int \rho^2\, dM = I_0\omega$$

From Art. 130,

$$Fd = I\alpha = I\frac{d\omega}{dt}$$

$$\int d \times F\, dt = \int_{\omega_0}^{\omega} I\, d\omega$$

$$\int d \times F\, dt = I\omega - I\omega_0$$

The statement of this relation is as follows:

In any motion of rotation, the sum of the angular impulses of the forces acting is equal to the increase in angular momentum.

Both angular impulse and angular momentum are vector quantities and are represented graphically by vectors parallel to their axes of rotation. The convention for sign is the same as that used in connection with couples, as explained in Art. 31. If viewed so that the rotation of the angular impulse or momentum appears negative (clockwise), the vector points away from the observer. The vectors for Fig. 458 would be perpendicular to the plane of the figure, and the arrows would point toward the observer, since the rotation is counterclockwise. Like other vector quantities, angular impulse and angular momentum may be resolved into components or combined into resultants as desired.

EXAMPLE

A flywheel weighs 3200 lb. and is rotating at 125 r.p.m. Its radius of gyration is 3.3 ft., and its shaft is 6 in. in diameter. If the coefficient of friction f at the bearing is 0.02, how many revolutions will it make before coming to rest?

Solution.

$$\text{Friction} = 3200 \times 0.02 = 64 \text{ lb.}$$

The impulse of the friction $= 64t$

The angular impulse $= 64t \times \frac{1}{4} = 16t$

$$\text{Angular impulse} = \text{change in angular momentum}$$
$$16t = I\omega$$
$$16t = \frac{W}{g}k^2\omega = \frac{3200}{32.2} \times 3.3^2 \times \frac{125}{60} \times 2\pi$$
$$16t = 14{,}180$$
$$t = 885 \text{ sec.} = 14.75 \text{ min.}$$

The average speed of the flywheel is 62.5 r.p.m., so in 14.75 min. it will turn through $62.5 \times 14.75 = 922$ rev.

Problems

1. When another lubricant was used, the wheel referred to in the example came to rest in 18 min. 37 sec. Compute its coefficient of friction.

Ans. $f = 0.0159$.

2. A cast-iron disk 3 ft. in diameter and 2 in. thick is supported in bearings by means of a shaft 2 in. in diameter. The coefficient of friction $f = 0.02$. If a force of 12 lb. is applied vertically downward to a cord wrapped around the cylinder, what is the rim velocity 10 sec. after it starts from rest?

Ans. 13.83 ft./sec.

3. If in Prob. 2 the force is released at the end of 10 sec., how long will the disk rotate until the friction of the bearings brings it to rest?

Ans. 3 min. 14 sec.

175. Conservation of Angular Momentum.—If during any time t there is no external angular impulse on a body or system of bodies with respect to any given axis, the angular momentum with respect to that axis remains constant, irrespective of mutual actions and reactions between the bodies or parts of bodies. Since the internal forces always occur in pairs of equal and opposite forces during the same time t, the impulse of each pair $\int F\,dt - \int F\,dt$ reduces to zero. Since the angular impulse is zero, there can be no change in angular momentum.

Consider as an example two disks A and B, Fig. 459, supported on a horizontal shaft. Let disk A be fastened to the shaft and be at rest, while disk B rotates upon the shaft with angular velocity ω. Let the moment of inertia of disk B be I, and the moment of inertia of the entire system be I'. Since disk A is at rest, the angular momentum of the system is equal to the angular momentum of B, or $I\omega$. If, now, by some means the two disks are fastened together, as by allowing a bolt in B to drop into a hole in A, the two will rotate together with a new angular velocity ω'. The angular momentum of the system is now $I'\omega'$; and, since it has

not been changed by the internal action and reaction of the bolt and the disks,

$$I'\omega' = I\omega, \text{ so } \omega' = \frac{I}{I'}\omega$$

Both $I\omega$ and $I'\omega'$ are represented by the same vector as shown at the left of the figure.

As in the case of linear momentum, there is a loss of kinetic energy due to the mutual action between the two disks.

As another case, consider the two bodies A and B, Fig. 460, which may be moved along a horizontal axis and which are rotat-

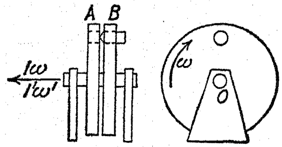


FIG. 459.

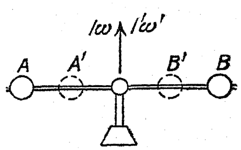


FIG. 460.

ing with their support about a vertical axis with angular velocity ω. If I is their moment of inertia with respect to their axis of rotation, their angular momentum is $I\omega$. If, now, by some internal action the bodies are displaced, as to A' and B', the moment of inertia of the system becomes I'. Then, as above, $\omega' = \frac{I}{I'}\omega$. Since I' is less than I, ω' must be correspondingly greater than ω.

Problems

1. In Fig. 459, the disks are steel, 2 ft. in diameter. Disk B is 2 in. thick, and disk A is 0.8 in. thick. If disk B is rotating at 90 r.p.m. and disk A is at rest, what will be their speed of rotation after the two are connected? What will be the loss in kinetic energy? *Ans.* 64.2 r.p.m.; 51 ft.-lb.

2. In Fig. 460, let A and B be cast-iron spheres 6 in. in diameter supported on a solid steel rod 5 ft. long and 1 in. in diameter. In positions A and B, the spheres are 4 ft. apart center to center and are rotating with the rod about the vertical axis at a speed of 60 r.p.m. By means of a stretched spring connecting them, the spheres are brought to positions A' and B', 1 ft. center to center. Neglecting the mass of the spring, compute the new speed of rotation. Compute the change in kinetic energy. Explain the gain. *Ans.* 354 r.p.m.; 766 ft.-lb. gain.

176. Resultant of Angular Momenta. *Gyroscope.*—In this discussion, only the case of gyroscopes with mutually rectangular axes will be considered. If the wheel shown in Fig. 461 is rotating a large angular velocity ω about the horizontal axis AB

which is supported only at point A, the axle being free to rotate in any direction about A, the wheel will not fall as it would if it were not rotating but will rotate, or *precess*, about the vertical axis through A. In Fig. 462, vector ON in the positive X direction represents the angular momentum $I\omega$ of the wheel about axis AB or OX, its spin axis. The external forces consisting of the weight W and the vertical reaction at A have a torque T

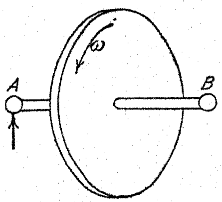


FIG. 461.

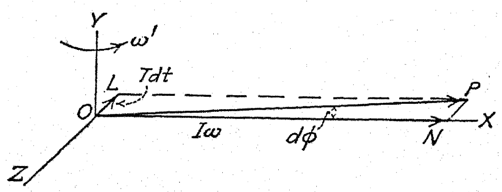


FIG. 462.

about axis OZ in the clockwise direction, and in time dt this torque generates an angular momentum Tdt about axis OZ. Since this torque is clockwise, the angular momentum is negative and is represented by vector OL. The resultant of these two angular momentum vectors is OP, at an angle $d\phi$ with the vector ON. This is the new axis of rotation, and, in order that it may become so, the axle AB rotates, or precesses, counterclockwise in the horizontal plane with a constant angular velocity ω'.

$$OL = ON\, d\phi$$
$$T\, dt = I\omega\, d\phi$$
$$T = I\omega \frac{d\phi}{dt}$$

But

$$\frac{d\phi}{dt} = \omega'$$

so

$$\mathbf{T} = \mathbf{I}\boldsymbol{\omega}\boldsymbol{\omega}'$$

Let the X and Z axes rotate with the wheel about the Y axis. Since the torque about axis OZ remains constant, the angular velocity of precession ω' will be constant if the velocity of spin ω is constant. As ω decreases on account of friction and air resistance, ω' increases.

If a couple is applied in the XZ plane to hurry or increase the velocity of precession around the Y axis, the rotating disk and axle will rise. The vector of the couple to hurry the precess counterclockwise, viewed from above, will point upwar

when combined with the spin vector will raise the outer end of the latter.

Similarly, a couple applied to retard the velocity of precession will cause the rotating disk and axle to fall.

Just as a torque about an axis normal to the spin axis will cause a precession about a third axis normal to the two, a forced precession about an axis normal to the spin axis will develop a torque about the third rectangular axis. For example, the driving wheels of a locomotive when going around a horizontal curve are forced to precess about a vertical axis at the center of curvature. This forced precession causes a torque about the axis tangent to the curve, so that the reaction of the outer rail on the driving wheel is greater than it is when on a straight track, and that of the inner rail is less.

Assume first a curve to the left. The spin vector of the wheels points to the left, toward the center of the curve. After the locomotive has moved a short distance around the curve, the spin vector points to the left and backward. The angular impulse vector, therefore, points backward and is produced by a counterclockwise torque viewed from the rear, a torque given by a heavier pressure on the right, or outer, wheel and a lighter pressure on the inner, or left, wheel. If the curve is toward the right, the spin vector still points to the left, away from the center of the curve. After the locomotive has moved a short distance around the curve, the spin vector points to the left and forward. The angular impulse vector therefore points forward and is produced by a clockwise torque viewed from the rear, a torque given by a heavier pressure on the left, or outer, wheel and a lighter pressure on the right, or inner, wheel.

If an ordinary top is spinning about its vertical geometric Y axis in a clockwise direction viewed from above, and a pressure in the positive X direction is applied at its upper part, the axis will not dip in the positive X direction but in the positive Z direction. The spin vector is vertical, directed downward. The torque vector is directed negatively along the Z axis. The resultant of these two vectors is directed downward and back, so the spin axis dips forward at the upper end.

If the top is spinning counterclockwise as viewed from above, the same torque will cause the upper end to dip in the negative Z direction.

EXAMPLE

The armature of the motor of an electric car weighs 600 lb. and rotates in the direction opposite to the rotation of the car wheels. The distance between bearings is 2 ft. and the radius of gyration of the armature is 6 in. The motor is geared so that it makes four revolutions to one revolution of the car wheels. The diameter of the car wheels is 33 in. If the car is going forward around a curve of 100 ft. radius with a velocity of 20 ft./sec., what are the pressures on the bearings if the center of the curve is to the right?

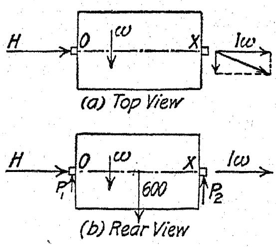


FIG. 463.

Solution.—Figure 463(*a*) is a top view, and Fig. 463(*b*) is a rear view of the motor. OX is the spin axis, a vertical axis through the center of curvature of the track is the precession axis, and any axis normal to these is the torque axis. Since the armature is being accelerated toward the center of the curve with an acceleration $a = v^2/r = 4$ ft./sec.^{2}, the horizontal pressure of the bearing

$$H = \frac{600}{32.2} \times 4 = 74.5 \text{ lb.}$$

This is due to the centrifugal force and would be the same if the motor were not rotating. Since there is no torque about the axis of precession, there are no horizontal components of the reactions at the ends of the armature normal to H.

Since the spin is backward, the angular momentum vector $I\omega$ points to the right. In order to combine with this momentum vector so as to produce precession to the right, the angular impulse vector must point backward along the track toward the observer. The torque to give the vector this direction is counterclockwise, so P_2 must be larger than P_1.

Since $T = I\omega\omega'$, by moments about O,

$$(P_2 \times 2) - (600 \times 1) = I\omega\omega'$$

$$I = \frac{W}{g}k^2 = \frac{600}{32.2} \times \frac{1}{4} = 4.66$$

The angular velocity of rotation of the car wheels is

$$\omega_1 = \frac{v}{r} = \frac{20}{1.375} = 14.55 \text{ rad./sec.}$$

Since the gear ratio is 4:1, the angular velocity ω of the armature is

$$4 \times 14.55 = 58.2 \text{ rad./sec.}$$

$$\omega' = \frac{v}{r} = \frac{20}{100} = 0.2 \text{ rad./sec.}$$

$$2P_2 - 600 = 4.66 \times 58.2 \times 0.2$$

$$2P_2 - 600 = 54$$

$$P_2 = 327 \text{ lb.}$$

Since $P_1 + P_2 = 600$,

$$P_1 = 273 \text{ lb.}$$

It is seen that the pressure on the bearing on the inside of the curve is 27 lb. heavier than if the motor were not rotating, and the pressure on the outside is 27 lb. lighter.

For a curve in the opposite direction, the angular momentum vector would still point to the right, and the angular impulse vector would have to point forward in order that the two might combine to produce precession to the left. The torque is therefore of opposite sign, so the heavier pressure is again on the inside bearing.

For a motor geared so that it rotates in the same direction as the car wheels, the heavier pressure is on the bearing on the outside of the curve.

Problems

1. In Prob. 1, Art. 116, it was found that a pair of 33-in. cast-iron car wheels weighing 700 lb. had a moment of inertia of 6.99 with respect to the axis of rotation. If a car is running at a speed of 45 m.p.h. around a 6° curve what is the extra pressure on the outer rail due to gyroscopic motion? Use 4.9 ft. as the distance from center to center of rails.

Ans. 4.7 lb. for each pair of wheels.

2. Describe the gyroscopic action of the flywheel of an automobile engine when rounding a curve (1) to the right; (2) to the left.

Ans. (1) Heavier pressure on rear bearing.

3. If an ordinary top is rotating clockwise viewed from above, and the upper end of the axis is pushed horizontally north, which way will it really lean? Explain. *Ans.* East.

4. If an airplane that is coming down head on at a steep angle changes direction by a short curve into the horizontal, what will be the gyroscopic action if the propeller is rotating clockwise when viewed from the rear?

Ans. Left end will be thrown forward.

5. A small gyroscope consists of a wheel with a heavy rim weighing 1 lb. and with a radius of gyration of 2 in. The wheel is rotating about its axis at 500 r.p.m. If placed with its axis horizontal and one end supported on a pivot about which it is free to precess, what is its speed of precession if the pivot is 1.5 in. from the center of gravity of the wheel?

Ans. 2.77 rad./sec.

6. A pair of locomotive driving wheels and their axle weigh 4500 lb. Their diameter is 6 ft., and their radius of gyration is 2 ft. Compute the added pressure due to gyroscopic action when the locomotive travels around a 4° curve at a speed of 60 m.p.h. The gage of the track is 5 ft. *Ans.* 202 lb.

177. Reaction of a Jet of Water.—If a jet of water of cross-sectional area A issues from the side of a vessel of water under head h, as shown in Fig. 464, it will have a velocity $v = \sqrt{2gh}$. The water before leaving the vessel is at rest, so the change in momentum in the direction of the jet is Mv per second, M being the mass of water flowing per second. $Mv = \frac{W}{g}v$, W being the weight of water flowing per second. If w is the weight of the unit volume of water, $W = wAv$. Then the change in momen-

tum per second is $Mv = wAv^2/g$. The change in momentum in time t is $\frac{wAv^2}{g}t$, and this must be caused by an impulse Ft. Then since

$$Ft = \frac{wAv^2}{g}t$$

$$\mathbf{F} = \frac{\mathbf{wAv^2}}{\mathbf{g}} = \mathbf{2wAh}$$

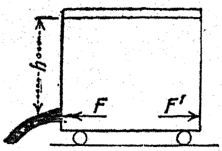


FIG. 464.

F' is the equal and opposite reaction of the water upon the vessel. If the vessel is not held by an external force, it will move to the right under the action of force F'. This principle is applied in the construction of rotating lawn sprinklers.

Problems

1. A cubical vessel 1 ft. on each side, weighing 7.5 lb., is full of water and is suspended from a cord so that its center of gravity is 5 ft. from the support. If a jet ¾ in. in diameter under a head of 6 in. is issuing from the middle of one side, how far from the vertical is the center of gravity displaced? *Ans.* 0.164 in.

2. Find the force necessary to hold the nozzle of a fire hose 1.5 in. in diameter discharging water under a head of 150 ft. *Ans.* 230 lb.

178. Pressure Due to a Jet of Water on a Vane.—If a jet of water is discharged perpendicularly against a stationary flat vane, as in Fig. 465, all the velocity of the jet in the original direction

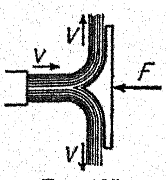


FIG. 465.

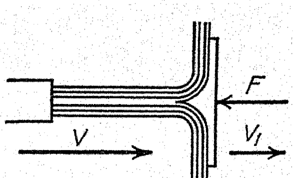


FIG. 466.

is destroyed. If W is the weight of water flowing per second, the change in momentum in the direction of the jet is Wv/g per second. The change in momentum in time t is Wvt/g, and this must equal the impulse of the force F which supports the vane. Then

$$Ft = \frac{Wv}{g}t = \frac{wAv^2}{g}t$$

$$\mathbf{F} = \frac{\mathbf{Wv}}{\mathbf{g}} = \frac{\mathbf{wAv^2}}{\mathbf{g}} = \mathbf{2wAh}$$

It will be noticed that the pressure P due to an impinging jet is twice as great as the static pressure $P' = wAh$ on the same

that the pressure each exerts upon the other is also along this line, the impact is called *direct central* impact. All other impacts are oblique.

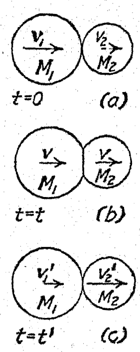


FIG. 469.

For simplicity assume the colliding bodies to be spheres, as in Fig. 469. The mass M_1 moving with velocity v_1 overtakes mass M_2 moving with velocity v_2. When the bodies first touch, as in Fig. 469 (*a*), the pressure between them is zero. For a short period of time, the centers approach each other, and each is deformed by the pressure of the other. When the pressure becomes a maximum, the deformation is a maximum, as shown in Fig. 469(*b*), and the bodies are moving with the same velocity v. If the bodies are inelastic, the pressure drops directly to zero, the deformation remains, and the two bodies go on together with velocity v. By the principle of conservation of linear momentum (Art. 173),

$$\mathbf{M}_1\mathbf{v}_1 + \mathbf{M}_2\mathbf{v}_2 = \mathbf{M}_1\mathbf{v} + \mathbf{M}_2\mathbf{v}$$

If the bodies are partially elastic, the pressure decreases gradually to zero, the original form is partially regained, and the two bodies separate, M_1 moving with velocity v_1' and M_2 with velocity v_2', as shown in Fig. 469(*c*). In this case also, by Art. 173,

$$\mathbf{M}_1\mathbf{v} + \mathbf{M}_2\mathbf{v} = \mathbf{M}_1\mathbf{v}_1' + \mathbf{M}_2\mathbf{v}_2'$$

The first period of time is called the *period of compression.* The second is called the *period of restitution.*

From the two equations above, the sum of the momenta before impact equals the sum of the momenta after impact.

$$\mathbf{M}_1\mathbf{v}_1 + \mathbf{M}_2\mathbf{v}_2 = \mathbf{M}_1\mathbf{v}_1' + \mathbf{M}_2\mathbf{v}_2'$$

The direction of v_1 should always be considered positive. If v_2 is in the opposite direction, it must be used as negative. The signs of v_1' and v_2' show their directions.

In the equations above, the value of g may be factored out, and the last equation may be written

$$W_1v_1 + W_2v_2 = W_1v_1' + W_2v_2'$$

The relative velocity of the two bodies before impact is $v_1 - v_2$, and the relative velocity after impact is $v_2' - v_1'$. Owing to the

fact that physical bodies are not perfectly elastic, the relative velocity after impact is always less than that before impact, and the ratio of the two is called the *coefficient of restitution*, represented by e.

$$e = \frac{v_2' - v_1'}{v_1 - v_2}$$

This may be written

$$\mathbf{e}(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{v}_2' - \mathbf{v}_1'$$

The value of e is zero for entirely inelastic bodies and would be unity for perfectly elastic bodies. The following table gives the values of e for several materials as determined by experiment:

Material	e	Material	e
Glass	0.95	Cast iron	0.50
Ivory	0.89	Lead	0.15
Steel	0.55		

If a body falls freely and strikes a fixed base, $v_2 = 0$, and $v_2' = 0$.

$$ev_1 = -v_1'$$

Let h be the height from which the body falls, and h' the height to which it rebounds. Since $v = \sqrt{2gh}$

$$e\sqrt{2gh} = -\sqrt{2gh'}$$
$$e^2h = h'$$

The kinetic energy lost in heat of impact may be found by subtracting the final kinetic energy of the two bodies from their initial kinetic energy.

If the impact of two bodies is oblique, the velocity of each body may be resolved into two components, one along the line of centers, the other normal to the line of centers. The latter component of each is unchanged by the impact. The former is changed the same as in direct central impact, and the final components are recombined to give the final velocities.

EXAMPLE 1

An inelastic body weighing 5 lb. is moving with a velocity of 10 ft./sec. and collides with another weighing 2 lb. moving in the opposite direction

ith 6 ft./sec. Compute the final velocity of the two bodies nd kinetic ene y due to the impact.

Solution.

$$W_1v_1 + W_2v_2 = W_1v + W_2v$$
$$(5 \times 10) - (2 \times 6) = (5 + 2)v$$
$$v = 5.43 \text{ ft./sec.}$$

$$\text{Initial K.E.} = \left(\frac{1}{2} \times \frac{5}{32.2} \times 100\right) + \left(\frac{1}{2} \times \frac{2}{32.2} \times 36\right) = 8.88 \text{ ft.-lb.}$$

$$\text{Final K.E.} = \frac{1}{2} \times \frac{7}{32.2} \times 5.43^2 = 3.21 \text{ ft.-lb.}$$

$$\text{Loss in K.E.} = 8.88 - 3.21 = 5.67 \text{ ft.-lb.}$$

EXAMPLE 2

A 4-lb. steel hammer with a horizontal velocity of 12 ft./sec. strikes a 10-lb. steel ball which is at rest. Get the velocity of each after the impact and the loss in kinetic energy.

Solution.—The hammer is W_1 and the ball is W_2. From the foregoing table, $e = 0.55$.

$$0.55(12 - 0) = v_2' - v_1' = 6.6$$
$$48 + 0 = 4v_1' + 10v_2'$$
$$v_1' = -1.29 \text{ ft./sec.}$$
$$v_2' = 5.31 \text{ ft./sec.}$$

The ball is driven in the direction the hammer was going originally; the hammer itself rebounds.

$$\text{Initial K.E.} = \frac{1}{2} \times \frac{4}{32.2} \times 144 = 8.95 \text{ ft.-lb.}$$

$$\text{Final K.E. of hammer} = \frac{1}{2} \times \frac{4}{32.2} \times 1.29^2 = 0.10 \text{ ft.-lb.}$$

$$\text{Final K.E. of ball} = \frac{1}{2} \times \frac{10}{32.2} \times 5.31^2 = 4.39 \text{ ft.-lb.}$$

$$\text{Loss in K.E.} = 8.95 - 4.49 = 4.46 \text{ ft.-lb.}$$

Problems

1. An inelastic body weighing 10 lb. is moving with a velocity of 8 ft./sec. and overtakes another weighing 15 lb. which is moving in the same direction with a velocity of 3 ft./sec. Get the final velocity and the loss in kinetic energy. *Ans.* 5 ft./sec.; 2.33 ft.-lb.

2. An inelastic body weighing 20 lb. and moving with a velocity of 30 ft./sec. strikes another inelastic body weighing 25 lb. which is at rest. Get the final velocity and the loss in kinetic energy. *Ans.* 13.33 ft./sec.; 156 ft.-lb.

3. Solve Prob. 2 if the 25-lb. body is mov toward the other with a velocity of 30 ft./sec. *Ans* t./sec.; 621 ft.-lb.

4. A tempered-steel ball drops from a height a fixed tempered-steel base and rebounds to a height of 1.98 ft. Compute the value of e. *Ans.* $e = 0.995$.

5. With what velocity must a tempered-steel hamm 1 strike a tempered-steel ball weighing 0.2 lb. wh h is at rest dri it with a velocity of 200 ft./sec.? Assume the lue of e giv in Prob. 4.
Ans. 20.3 ft./sec.

GENERAL PROBLEMS ON IMPULSE, MOMENTUM, AND IMPACT

1. A freight car weighing 60,000 lb. starts from rest and runs down a 1 per cent grade. If train resistance is 8 lb./ton, what is its velocity at the end of 2 min.? *Ans.* 23.18 ft./sec.

2. If the train resistance on the car described in Prob. 1 is $8 + 0.01t$ lb. per ton, t being in seconds, what is its velocity at the end of 2 min.?
Ans. 22.02 ft./sec.

3. If at the end of 2 min., brakes are applied so as to stop the car described in Prob. 1 in 15 sec., what is the braking force required? *Ans.* 3240 lb.

4. A rifle weighing 8 lb. shoots a bullet weighing 0.1 oz. with a muzzle velocity of 2500 ft./sec. Get the recoil velocity of the rifle.
Ans. 1.953 ft./sec.

5. A push car weighing 300 lb. is moving with a uniform velocity of 12 ft./sec. If a man weighing 160 lb. boards it from the side with no velocity in the direction in which the car is moving, what is their velocity after the man comes to rest with respect to the car? *Ans.* 7.83 ft./sec.

6. A mine cage weighing 800 lb. is hung from the drum of a hoisting engine which weighs 1200 lb. The diameter of the drum is 5 ft., and its radius of gyration is 2 ft. Neglecting axle friction, get the velocity that the cage will attain if allowed to fall for 3 sec. before the brake is applied.
Ans. 49.3 ft./sec.

7. A flywheel weighing 420 lb. is 3 ft. in diameter and has a radius of gyration of 1.2 ft. If placed on a 15° plane and released to roll down, what is its linear velocity after 5 sec.? What is the frictional force of the plane on the wheel? *Ans.* 25.4 ft./sec.; 42.5 lb.

8. Two cast-iron spheres, each 4 in. in diameter and each with a ½-in. hole through the center, are free to slide on a horizontal steel rod ½ in. in diameter and 2 ft. long with a stop at each end. The rod is free to rotate about a vertical axis at its center. In their original position, the center of each sphere is 3 in. from the axis, and the spheres are rotating with the rod about the vertical axis at 360 r.p.m. If the spheres are released on the rod and allowed to slide out until they strike the stops at the ends of the rod, what is the final speed of rotation? What is the loss in kinetic energy? Neglect the mass of the stops and the rotating vertical axle.
Ans. 49.2 r.p.m.; 32.4 ft.-lb.

9. A pair of locomotive driving wheels 6 ft. in diameter weigh 6000 lb. and have a radius of gyration of 2.5 ft. Compute the extra gyroscopic pressure of the outer rail on the wheel as the locomotive travels around an 8° curve at a speed of 45 m.p.h. The rails are 60 in. from center to center. Neglect the superelevation *Ans.* 472 lb.

10. A gyroscope consists of a circular rim weighing 16 lb. on a bicycle wheel 28 in. in diameter. The wheel is on the end of a horizontal axis 3 ft. long, supported at its middle on a pivot so that it is free to rotate in any

direction. If the wheel is rotating about its own axis with a speed of 800 r.p.m. clockwise when viewed from the pivot, in which direction will it precess, and with what angular velocity?

Ans. Counterclockwise; $\omega' = 0.425$ rad./sec.

11. Discuss the gyroscopic action of the propeller of a ship as the ship pitches fore and aft on the waves.

12. Show that if the propeller of an airplane is rotating clockwise viewed from the rear, a quick left turn tends to raise the nose of the airplane and that a vertical loop tends to turn the nose to the right.

13. In certain tests of airplanes, it is required that the airplane shall come down head on with open throttle from a height great enough so that terminal velocity is attained, then that it shall be flattened out into the horizontal in a curve short enough to produce a normal acceleration $a_n = 9g$. At a speed of 300 m.p.h., what is the radius of the required curve? If the propeller weighs 340 lb. with a moment of inertia of 8.19 and is rotating at 1445 r.p.m., what is the amount of the gyroscopic couple caused?

Ans. 670 ft.; 814 lb.-ft.

14. If the coefficient of restitution $e = 0.9$ for a rubber ball, how high will it rebound if dropped from a height of 8 ft.? *Ans.* 6.48 ft.

15. A small cast-iron sphere when dropped from a height of 16 in. upon a cast-iron block rebounded 5.2 in. Compute e. *Ans.* $e = 0.57$.

16. A 100,000-lb. railway car moving with a velocity of 4 m.p.h. overtakes and collides with another weighing 80,000 lb. moving with a velocity of 2 m.p.h. If $e = \frac{1}{4}$, what is the loss in kinetic energy?

Ans. 5450 ft.-lb.

17. From a point 5 ft. above the ground, a ball is thrown upward at an angle of 60° with the horizontal against a wall 25 ft. distant. If its initial velocity is 60 ft./sec., and e for the ball is 0.4, where and with what velocity will the ball strike the ground?

Ans. 29.9 ft. from wall; $v = 56.4$ ft./sec.; 12°20′ with the vertical.

18. If the ball described in Prob. 17 is thrown horizontally with the same velocity but at an angle of 75° with the wall, where will it strike the wall? Where will it strike the ground, and with what velocity?

Ans. 6.7 ft. along wall, 2.0 ft. from ground; 8.65 ft. along wall, 2.9 ft. from wall; $v = 33.2$ ft./sec.; 32°45′ with ground, 56°10′ with wall.

INDEX